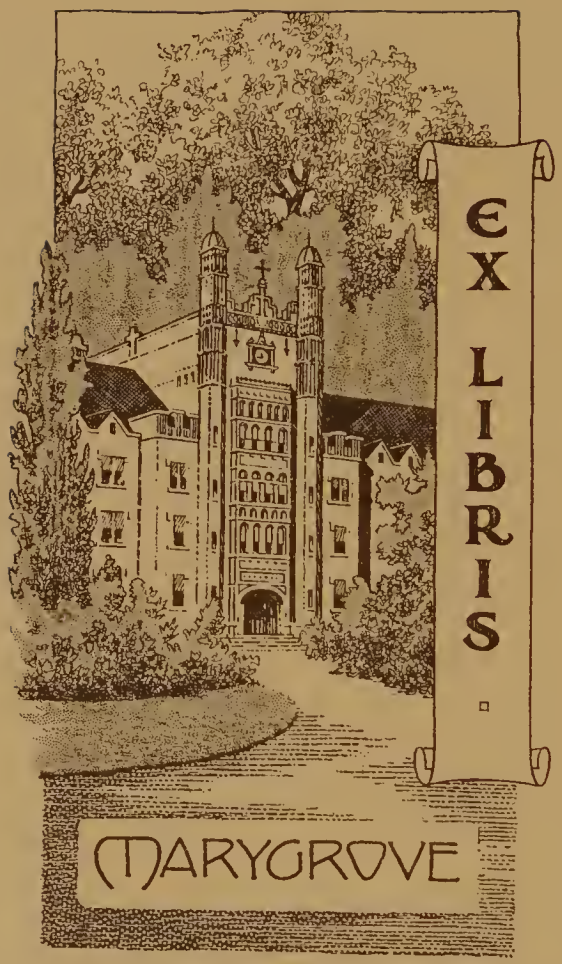
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DESCRIPTIVE GEOMETRY

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DESCRIPTIVE GEOMETRY

BY

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FOREWORD

TO STUDENTS OF ENGINEERING

ENGINEERS and architects are constantly confronted with the necessity of determining the relative directions of lines and planes, the values of angles, the lengths of lines, the intersections of planes and surfaces, the true shapes and sizes of portions of planes, and many more similar problems. All of this is in connection with their necessary dealing with three-dimensional orthographic projections or drawings of objects met in their respective special fields; architecture, and architectural, civil, electrical, mechanical, mining, aeronautical and other branches of engineering. These problems are matters of everyday occurrence and must be solved. Descriptive Geometry is the special form of mathematics or geometry that is employed in solving them. Of necessity it is closely related to Solid Geometry since it deals with three-dimensional geometry, and a thorough understanding of Solid Geometry is necessary to a mastery of Descriptive Geometry.

It is hardly possible to say when the principle of Orthographic projection was devised. By this term we mean the principle of projecting any element of an object upon a plane by means of lines perpendicular to that plane. It seems, however, that the graphic methods of solving problems like those outlined above were first used and later discussed comprehensively by the French mathematician and military engineer, Gaspard Monge. Their great value was quickly recognized and the subject was rapidly explored and developed. The first treatise on the subject appeared in French in 1795. The science was introduced into American education in 1816 by the French engineer Crozet. The early American texts on the subject were published by Crozet in 1821, Davies in 1826, Warren in 1860 and Church in 1865. The treatise prepared by Professor Church stood without a serious rival for more than forty years.

The rapid expansion which has taken place in every form of commerce since the beginning of this century has resulted in

demands on the engineer which he has been able to meet only by a revamping of all of his methods. This has been reflected in engineering education and within the past two decades the literature of engineering education has been almost entirely remade. A host of new books have appeared on the subject of Descriptive Geometry, each new author sincerely endeavoring to solve some of the problems of the increased and ever-increasing demands of engineering. Although these books differ greatly in the treatment of the subject, one fact is well established, that the engineer's need for a thorough acquaintance with the methods of solving the problems of his graphic language is as great as if not greater than ever before.

The subject of descriptive geometry has also come to be regarded as having special merits for the development of certain qualities which are essential to the engineer and are expected of him. These are the quality of vision, the ability to reason clearly and logically, and the ability to plan thoroughly and comprehensively. We all admire the athlete with his beautiful physique, great endurance and keenness of perception, and the artist with his skill to produce wonderful paintings or compose inspiring music. We know full well what labor each has expended in acquiring that skill, and that that is the only manner in which the skill can be acquired and kept. The royal road to success in engineering is likewise by the road of consistent hard labor, the products of which are those qualities which we have come to expect of the engineer. It is natural and right therefore that courses in descriptive geometry are deliberately planned and administered for the development of qualities as well as for the acquisition of knowledge.

The writer wishes to acknowledge his indebtedness to Professor J. C. Palmer of the University of Michigan for his valuable suggestions and assistance in this revision.

H. W. Miller.

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DESCRIPTIVE GEOMETRY

PART I

CHAPTER I

ABBREVIATIONS, NOTATION, CONVENTIONS, ETC.

GENERAL OBSERVATIONS

(1) Not infrequently students complain to their instructors of their inability to understand a text on Descriptive Geometry, and request advice concerning other texts that they may purchase to assist them. It is true that few students, upon reading for the first time what we term a "solution," understand it. But no number of auxiliary texts assist materially in any other manner than that which we hope our students will pursue with any text. It is necessary to re-read, reason, and ponder about any form of solution until, piece by piece, part by part, the mind finally builds up a picture of the problem and its elements. It is in this manner that what we term the ability to "visualize" is developed. We have said that this is a prime aim of the course, and it is not desired that the very thing we hope to accomplish be defeated by serving the exercises in any over-digested form. No student, and certainly no instructor, should desire it.

If one begins to complain of the incompleteness of the various portions of a text on this subject, it is a fairly clear

indication of a lack of thorough familiarity with Plane and Solid Geometry. It is not the purpose or function of this text to teach either. A thorough knowledge and familiarity with both must be assumed. Those who have difficulty in understanding statements herein are advised to familiarize themselves at once and thoroughly with such Geometry; it is hardly the proper function of a text on any subject to teach the prerequisites.

(2) It is important that each student make this chapter a constant handbook of reference; for, in the conscientious mastery of correct notation and conventions, lies half the victory in Descriptive Geometry. Carelessness in these matters, as well as general carelessness in methods of study and execution of the work, make a nightmare of indefiniteness and intangibility of Descriptive Geometry, which is one of the most beautifully definite subjects.

It is not desired that any student attempt to memorize the following symbols and abbreviations, but that he refer constantly to it for information on the proper lettering of points, designation of certain angles, and the character of line to be used for each different purpose. In a short time all these will be memorized from constant use, and Descriptive Geometry will have been made a far easier subject for the small amount of trouble.

ABBREVIATIONS

(3) H = Horizontal plane of projection.
V = Vertical " " "
P = Profile " " "
GL = Ground Line.
G_1L_1 = Profile Ground Line.

STANDARD CONVENTIONAL LINES

(4) Ground lines

Given lines
Auxiliary lines
Solid, medium weight.

Projection lines
Construction lines
1/16″ dash, 1/32″ space, medium weight.

Hidden lines
3/16″ dash, 1/16″ space, medium weight.

Required lines
Solid, heavy weight

Traces of Given planes
3/8″ dash and 1/24″ dot and space, medium weight.

Traces of Required planes
3/8″ dash and 1/24″ dot and space, heavy weight.

Axes of Solids
Axes of Revolution
1/2″ dash 1/24″ double dot and space, medium weight.

SYMBOLS

(5) $\perp$ = " perpendicular to "
$/\!/$ = " parallel to "
$/$ = " inclined to "
$\angle$ = " angle "
X = **Angle** between two intersecting **Lines.**
= " between two intersecting **Planes.**
= " between a **Line** and a **Plane.**
Θ, θ = " which any **Line** makes with H.
= " " " **Plane** " " H.
Φ, ϕ = " " " **Line** " " V.
= " " " **Plane** " " V.
Π, π = " " " **Line** " " P.
= " " " **Plane** " " P.

NOTATIONS

(6) a = H projection
a' = V "
a'' = P "
} of point A.

tTt' or HT, VT = traces of plane T.
Tt or HT = H trace of plane T.
Tt' or VT = V trace of Plane T.
$t''t''$ or PT = P trace of plane T.

h = H projection
h' = V "
h'' = P "
} of the H piercing point of any line.

v = H projection
v' = V "
v'' = P "
} of the V piercing point of any line.

p = H projection
p' = V "
p'' = P "
} of the piercing point of any line in either the P plane or any given oblique plane.

A_r =
B_r =
C_r =
} revolved positions of points.

SYSTEM OF COORDINATES

(7) A point on the ground line, called the **origin**, is used as the reference point for the system of coordinates by which the location of points is stated. The origin is a point common to three mutually perpendicular planes, the H plane, the V plane and a P plane. A point in space is designated by its distances from these three planes.

x = distance of point to right or left of P, as x is + or −.
y = distance of point above or below H, as y is + or −.
z = distance of point before or behind V, as z is + or −.

The coordinates are always given in this order and the distances are stated in inches; e.g., the point located by $(6, 3, -2\frac{1}{2})$ is 6 inches to the right of the P plane, 3 inches above H and $2\frac{1}{2}$ inches behind V.

The H and V traces of a plane are designated by coordinates and angles, as follows:

x = distance of the point of intersection of the traces with GL, to the right or left of the origin, as x is + or −
y = angle V trace makes with H, above or below H, as y is + or −.
z = angle H trace makes with V, before or behind V, as z is + or −.

Angles y and z are measured from the portion of GL to the right of the intersection of the traces.

When a plane is parallel to GL, distance x becomes infinite (∞) and angles y and z become 0°. In this case,

y = distance of V trace above or below H, as y is + or −.
z = distance of H trace before or behind V, as z is + or −.

LETTERING ON PLATES

(8) It is important that the projections of all points, lines and the traces of planes shall not only be lettered, but lettered carefully. The following alphabets are recommended:

abcdefghijklmnopqrstu
12345 vwxyyz 67890
ABCDEFGHIJKLMN
OPQRSTUVWXYZ

abcdefghijklmnopqrstu
12345 vwxyz 67890
ABCDEFGHIJKLMN
OPQRSTUVWXYZ&

IMPORTANT PROPOSITIONS IN SOLID GEOMETRY

(9) *a.* Two planes, if not parallel, intersect.

b. The intersection of two non-parallel planes is a right line.

c. If a straight line is perpendicular to a plane, every plane through the line is perpendicular to the plane.

d. If two parallel planes are intersected by a third plane the lines of intersection are parallel.

e. If a plane contains a line parallel to a given line, the plane is parallel to the given line.

f. If a line is perpendicular to a plane, every line in that plane passing through the piercing point is perpendicular to the given line.

g. The angle which a line makes with a plane is the angle the line makes with its projection on that plane.

h. If two planes are perpendicular to a third plane their line of intersection is perpendicular to that plane.

i. One point and a direction or two points determine a line.

j. Two intersecting lines, two parallel lines, three points, a point and a line, or a point and a relation to some known line or plane, determine a plane.

CHAPTER II

THE POINT AND LINE

(10) **Descriptive Geometry** is that branch of mathematics which has for its object the solution of all geometric problems which arise in representing objects or geometric magnitudes by drawings. These geometric magnitudes are represented by drawings that are known as **projections**, which are made on conventional intersecting planes, usually three, and commonly though not necessarily at right angles to each other.

COORDINATE PLANES OR PLANES OF PROJECTION

(11) The three conventional planes mentioned in the preceding article are named the **planes** of **projection,** and are: a Vertical plane, a Horizontal plane, and an End or Profile plane which is perpendicular to both the Vertical and Horizontal planes. These same planes are usually used as coordinate planes in stating the location of a point in space.

For convenience, we speak of the Vertical plane as the V plane, the Horizontal plane as the H plane, and the Profile plane as the P plane. These three planes are shown in their relative positions in Fig. 1.

GROUND LINE

(12) The line of intersection of the H and V planes is known as the **ground line** and for convenience is referred to

as GL. The intersection of the P plane with the V plane is known as the **profile ground line** and is referred to as G_1L_1.

The H and V planes form four dihedral angles or quadrants, which are named the 1st, 2nd, 3rd and 4th in the order shown in Fig. 1.

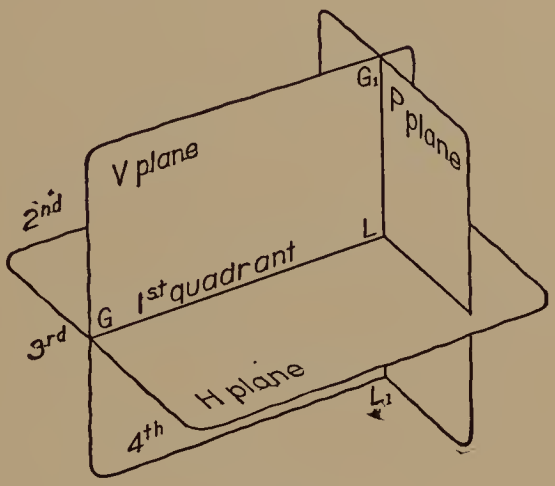


FIG. 1.

PROJECTIONS

(13) The representations of objects on the planes of projection are named **projections** and have the same appearance to the eye of an observer as the object itself when the observer is in a certain position. The choice of positions, named **points of sight** or **view points,** in which the observer may be placed, gives rise to two kinds of projections.

SCENOGRAPHIC PROJECTION OR PERSPECTIVE

(14) When the view point or the eye of the observer is at a finite distance, e.g., 100 ft. from the plane of projection, no two lines of sight from the view point to different points of the object are parallel and the projection is either *smaller* or *larger* than the object itself. Such projections are **sceno-**

graphic. The usual picture, photograph or painting is a **scenographic projection.**

ORTHOGRAPHIC PROJECTION

(15) In orthographic projection the points of sight or view points are always at infinite distances from the planes of projection; for convenience therefore the lines of sight from the view points to all points of an object, however large, are assumed to be parallel and perpendicular to the planes of projection. The point of sight for the projection on the horizontal plane (the H projection) is assumed at an infinite distance *above* the H plane; the view point for the projection on the V plane (the V projection) is assumed at an infinite distance in *front* of the V plane. For the purposes of normal representation, that is, commercial drafting, the view point for the profile plane is assumed at an infinite distance on the side of the profile plane opposite the object. For the purposes of descriptive geometry it may be at an infinite distance from the profile plane on either the side of the object or the opposite side. In either case, however, the projection is a representation of what can be seen from the view point. Thus to obtain the orthographic projections of an object on the H, V, and P planes, three points of sight are necessary. One of the projections (H) has the appearance of the top of the object, another (V) the front, and the third (P), the appearance of the side regardless of the dihedral angle in which the object is placed.

PROJECTING LINES

(16) The H projection of a point, for example, A, is the point in which the line of sight through A, pierces the H plane. The portion of the line of sight between the point and the plane of projection is named a **projecting line.** This line, as all lines, is actually indefinite in extent and the entire line may be considered the projecting line if desired. The projecting lines of a point may then be defined as the

lines passed through the point perpendicular to the planes of projection. Aa, Aa', and Aa'', Fig. 2, are projecting lines.

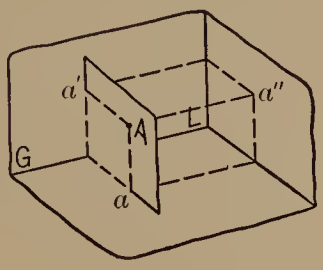


Fig. 2.

PROJECTIONS OF A POINT

(17) In summary of the preceding several articles, we may say that the H projection of a point A is the H piercing point of a perpendicular through A to the H plane. The V projection is the V piercing point of a perpendicular through A to the V plane. The P projection is the P piercing point of a perpendicular through A to the P plane. These projections are designated as in Fig. 2, the H projection a, V projection a', and P projection a''. The projections of all points are designated or lettered in a similar manner.

ALPHABET OF A POINT

(18) A point may be placed in nine different positions with respect to the planes of projection ; these nine positions constitute what is known as the **alphabet** of a **point.** These positions are respectively:

1 In the 1st quadrant.
2 In the 2nd quadrant.
3 In the 3rd quadrant.
4 In the 4th quadrant.
5 In the H plane before V.
6 In the H plane behind V.
7 In the V plane above H.
8 In the V plane below H.
9 In both H and V.

These positions are shown in perspective in Fig. 3.

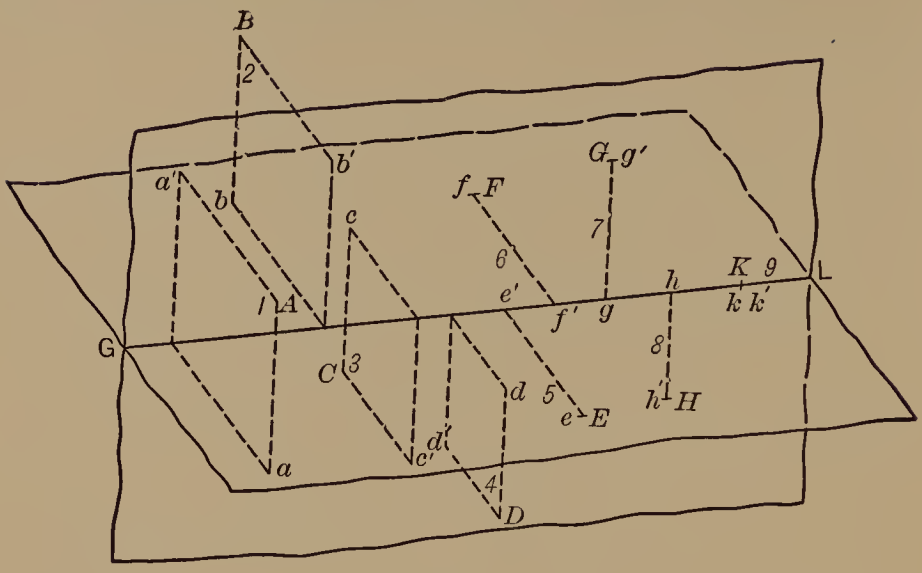


FIG. 3.

POINT DETERMINED BY ITS PROJECTIONS

(19) If through the H projection, a, of a point A, Fig. 4, a perpendicular be erected to the H plane, it will contain the point A; likewise a perpendicular to the V plane through the V projection, a', will contain point A; the point is hence determined, being the intersection of these two perpendiculars.

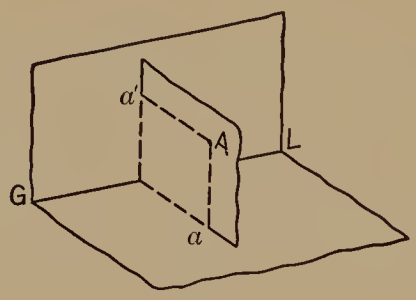


FIG. 4.

LOCATION OF POINTS

(20) If through the projecting lines Aa' and Aa. Fig. 4, a plane be passed, it will be perpendicular to both H and V

and intersect H and V in lines perpendicular to GL. These lines with Aa' and Aa form a rectangle in which the distance from a' to GL equals Aa and the distance from a to GL equals Aa'. From the above we deduce that:

a. The distance from a point to the H *plane is shown by the distance from its* V *projection to* GL.

b. The distance from a point to the V *plane is shown by the distance from its* H *projection to* GL.

c. If a point lies in V *its* H *projection is in* GL; *it cannot be in* GL *otherwise. Conversely, if the* H *projection of a point lies in* GL *the point lies in* V.

d. If a point lies in H *its* V *projection is in* GL; *it cannot be in* GL *otherwise. Conversely, if the* V *projection of a point lies in* GL *the point lies in* H.

REVOLUTION OF THE COORDINATE PLANES

(21) The question has probably already arisen as to how the projections on the three planes may be represented on a plane drawing. This is accomplished through the

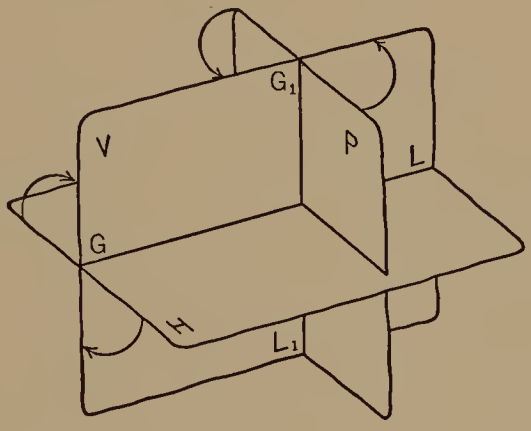


Fig. 5.

revolution of the H and P planes about their respective ground lines into coincidence with V. With GL, the line of

intersection of H and V, Fig. 5 as an axis, the part of H in front of V is revolved *down*, the part behind *up*, of course, until H coincides with V. Using G_1L_1 as an axis, the portion of P in *front* of V is revolved to the *right*, (or *left*), the portion *behind* to the *left*, (or *right*), until P coincides with V. The three planes now coincide.

In the revolution just explained, the V projections of points are not affected; i.e., the V projections of points in the 1st and 2nd quadrants are *above* GL, and those of points in the 3rd and 4th quadrants are *below* GL, regardless of any

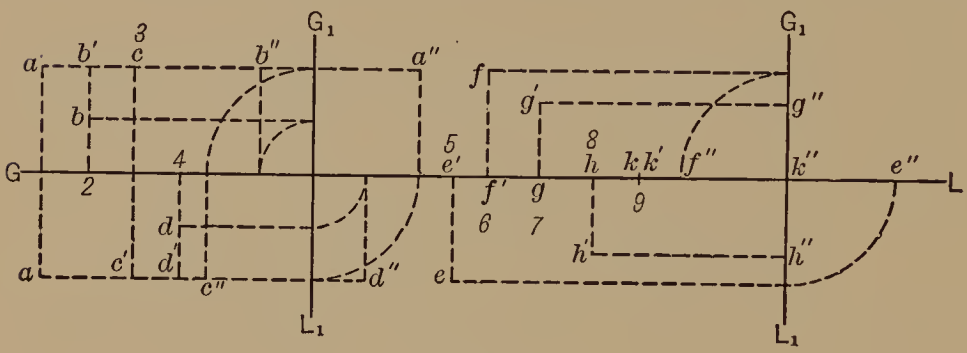


Fig. 6.

revolution of planes; the H projections of all points in *front* of V, however, either in the 1st or 4th quadrants, being in the portion of H which was revolved down, now come *below* GL; see Fig. 3, then Fig. 6. The H projections of all points *behind* V, in either the 2nd or 3rd quadrants, are found *above* GL since that part of H was revolved up.

As the H plane is revolved about GL, as an axis, the H projection of a point moves in a plane which is perpendicular to GL; after the revolution the H projection is therefore found on the perpendicular to GL, through the V projection of the point. *The* H *and* V *projections of a point are always on the same perpendicular to* GL.

The orthographic projections of the nine positions which constitute the *alphabet* of a point are shown in Fig. 6. (Compare Fig. 3.)

LINES

(22) Since a right line is generated by a point moving in some fixed direction, the H projection of a right line is made up of the H projections of the successive positions of the generating point. The H projecting lines of all of these points, being perpendicular to H, constitute a plane which is perpendicular to H; the line of intersection of this plane with H is the H projection of the given line. The H projection of a line may thus be considered a line of intersection with H of a plane passing through the given line and perpendicular to H. This auxiliary plane is named the **H projecting plane** of the line. The V and P projections of a right line are the lines of intersection with V and P of projecting planes through the line and perpendicular respectively to the V and P coordinate planes.

Since the H projection of a right line is a right line, the projection may be determined by passing perpendiculars through any two points of the line to H; the line joining the H piercing points of these perpendiculars, Fig. 7, is the required H projection. The V and P projections may be determined in a similar manner.

LOCATION OF A LINE

(23) If two projections of a line are given, the line is in general completely determined. For, if through the projections of any point of a line, e.g., b and b', Fig. 7, perpendiculars to the planes of projection be erected, these perpendiculars will intersect in the corresponding point, i.e., B, of the right line. The point A is similarly determined from a and a'. The right line joining these two points, A and B, is the required line, Fig. 7. Or again, if through the H projection we pass a plane perpendicular to H, and through the V projection a plane perpendicular to V, the line will lie in both of these planes, hence, is their line of intersection.

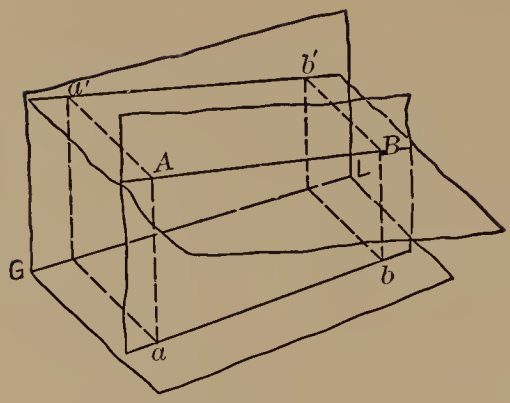


FIG. 7.

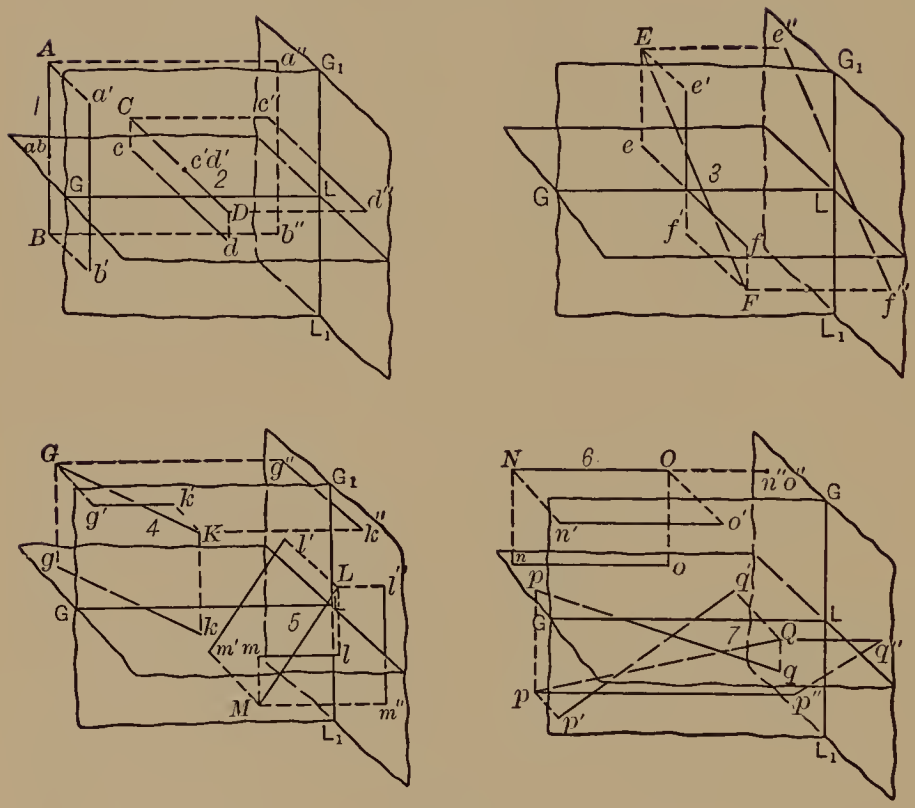


FIG. 8.

ALPHABET OF A LINE

(24) In determining the different positions in which a line may be placed with respect to the planes of projection, the quadrants are not considered and the lines are under-

stood to be indefinite in extent. The several positions constituting the alphabet are the *characteristic* positions in which the line may be placed. These positions are shown

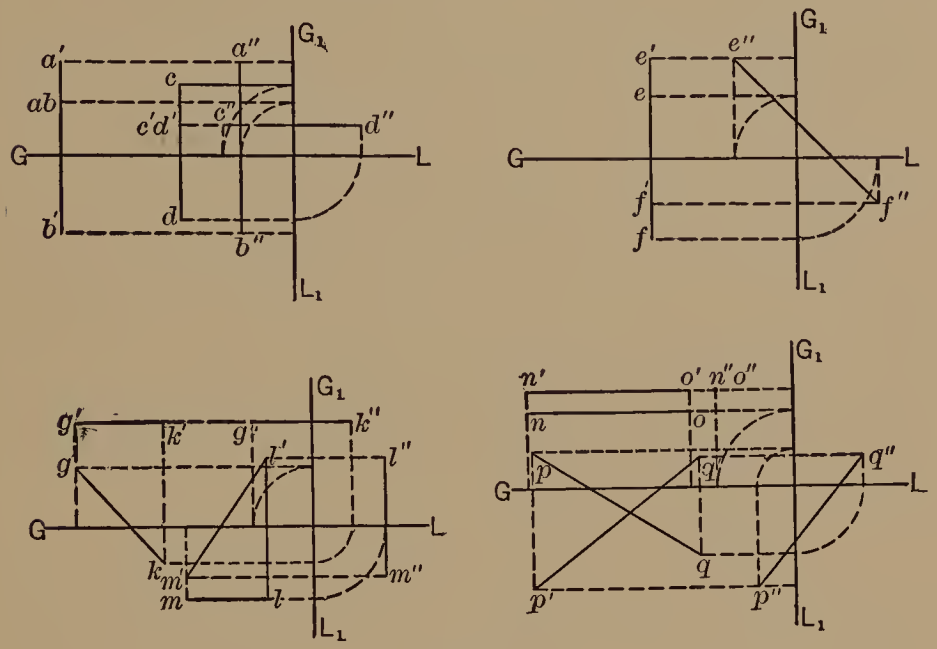


FIG. 9.

pictorially in Fig. 8. Fig. 9 shows the alphabet of the line in orthographic projection.

These positions of the line and corresponding projections are as follows:

1 Perpendicular to H	V projection a line perpendicular to GL. H projection a point.
2 Perpendicular to V	V projection a point. H projection a line perpendicular to GL.
3 Lying in a plane perpendicular to GL	Both projections perpendicular to GL.
4 Parallel to H and inclined to V	V projection parallel to GL. H projection inclined to GL.
5 Parallel to V and inclined to H	V projection inclined to GL. H projection parallel to GL.

6 Parallel to both H and V—Both projections parallel to GL.

7 Inclined to H, V, and P—Both projections inclined to GL.

No mention is made here of the profile relations but they may be easily understood from the figures.

H AND V PROJECTIONS OF POINTS AND LINES IN THE PROFILE PLANE

(25) When dealing with the H and V projections of any point or line which lies in the Profile plane, that plane is conceived to be in its original position, i.e., perpendicular to GL. In this case G_1L_1 represents both the intersection of P with V, and that of P with H; hence both the H and V projections of the point or line lie on G_1L_1.

PROFILE PROJECTIONS OF POINTS AND LINES

(26) When dealing with the Profile projections of points and lines, the Profile plane is considered as revolved about G_1L_1 until it coincides with V. In this case G_1L_1 represents the intersection of P and V, and GL represents the intersection of P and H; or for convenience, G_1L_1 may be considered the V plane, and GL, the H plane, viewed edgewise.

As explained in Art. 17, the P projection of a point A is obtained by passing a projecting line through A perpendicular to P; the piercing point of this perpendicular in P is the P projection of A. This projecting line, Fig. 2, Art. 17, pierces P at a distance above H equal to Aa, which in turn equals the distance from a' to GL. Likewise, it pierces P at a distance before V equal to Aa' which is equal to the distance from the H projection, a, to GL. Then, after revolving the Profile plane about G_1L_1 into V, Fig. 6, a'' is found above GL a distance equal to the distance from a' to GL, or on a line parallel to GL through a'; likewise, since A is in the 1st quadrant, a'' will be to the right of G_1L_1, a distance equal to the distance that the H projection,

a, is below GL. The construction for finding the Profile projections of points in the various quadrants, from their H and V projections, is shown in Fig. 10.

The profile projection of a line joining two points is the line joining the profile projections of those two points.

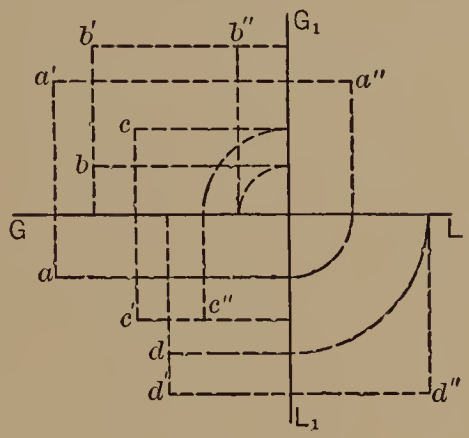


Fig. 10.

(27) Problem 1.—**To pass a line through a given point.**

Solution:

Since the projections of a line are made up of the projections of all points of the line, the projections of all lines which pass through a given point pass through the corresponding projections of the point. If, therefore, the two projections of a line pass through the corresponding projections of a point, the line passes through the point.

(28) Problem 2.—**To assume a right line.**

Solution:

A line may be assumed at will in any position and direction in space. The projections of that line may hence appear in any direction, provided each point of the one has a corresponding point on the other.

To assume a right line in the first quadrant the V projection is drawn in any position and in any direction above GL, Fig. 11; a' and b' are selected on this projection as the V projections of two points A and B of the line. Perpen-

diculars to GL are then passed through a' and b'; ab, in any position and direction below GL and joining these perpendiculars, is the H projection of a line AB in the first quadrant.

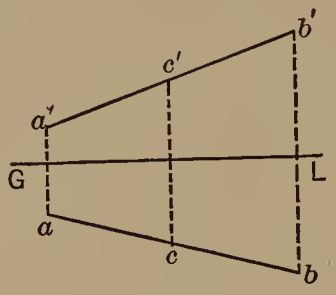


Fig. 11.

(29) Problem 3.—**To assume a point on a line.**

Solution:

The projections of a point of any line must lie on the corresponding projections of the line and on the same perpendicular to GL.

A point C may be assumed on line AB, Fig. 11, by assuming c' at will on the V projection of AB; a perpendicular to GL through c' intersects ab at c, which is the H projection of point C.

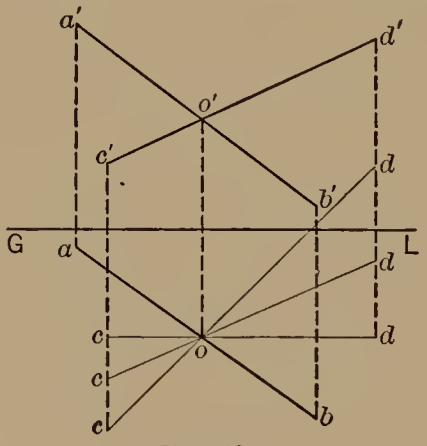


Fig. 12.

(30) Problem 4.—**To assume two intersecting lines.**

Solution:

If two lines intersect, their projections will intersect, and the

points of intersection of their projections will be the projections of their point of intersection.

Since the point of intersection is common to both lines, its V projection lies on the V projection of each line, hence is their intersection; the H projection of the point of intersection is likewise the intersection of the H projections. These two points, being the projections of the same point, lie on the same perpendicular to GL. Both projections of one line AB, Fig. 12, may therefore be assumed at will, and on them the projections of O, the point of intersection of the other line CD with AB. The H and V projections of CD are then drawn through the respective projections of O in any desired direction.

PARALLELISM

(31) Two lines are said to be parallel if they lie in the same plane and do not meet however far produced. However, for practical purposes and for clearness in analysis all parallel lines may be considered as meeting at infinity.

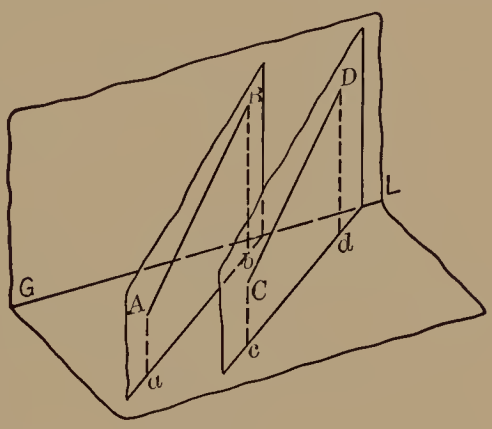


Fig. 13.

PARALLEL LINES

(32) If two lines are parallel their projections on any plane are parallel. Lines AB and CD, Fig. 13, are given parallel. The H projecting planes through these two parallel lines are parallel since the one contains two intersecting lines, AB and B*b*, parallel respectively to the two intersecting lines CD and D*d* of the other. We have then two parallel planes intersected by a third plane (the H plane). The lines of intersection, *ab* and *cd*, are therefore parallel. It is obvious

that the same reasoning will hold true for the V and P projections.

PERPENDICULAR LINES

(33) If two lines are perpendicular their corresponding projections are not necessarily perpendicular, but if one of the lines, or both, are parallel to a plane of projection, then their projections on that plane are perpendicular.

At any point on a line an infinite number of lines perpendicular to the given line may be constructed, their locus being a plane through the given point and perpendicular to the given line. Of this infinite number of perpendiculars, one must be parallel to H, another parallel to V, and a third parallel to P. Through a given point not on the line, only one perpendicular to the given line may be constructed and this perpendicular is not necessarily parallel to any plane of projection.

If the given line is oblique to H, V, and P, it follows from the above that, of the infinite number of perpendiculars at a given point on the line, there will be three cases in which one projection of the constructed perpendicular will be perpendicular to the corresponding projection of the given line. If two of these perpendiculars be constructed, they will determine the plane perpendicular to the given line and any other line in this plane and through the given point will be perpendicular to the given line.

The perpendicular from a point not on the line will not in general have any of its projections perpendicular to the corresponding projections of the given line. To construct the perpendicular in this case: pass a plane through the point and perpendicular to the given line; find the point in which the line pierces this plane; and join this piercing point with the given point.

If the given line is parallel to one of the planes of projection, any line perpendicular to it will have its projection on that plane perpendicular to the corresponding projection of the given line.

CHAPTER III

PLANES

REPRESENTATION

(34) Planes are represented on the coordinate planes of projection by their *lines of intersection* with these planes. These lines of intersection are called the **traces** of the plane; e.g., the line of intersection of a plane T with V is called the V trace of T, the intersection of T with H is called the H

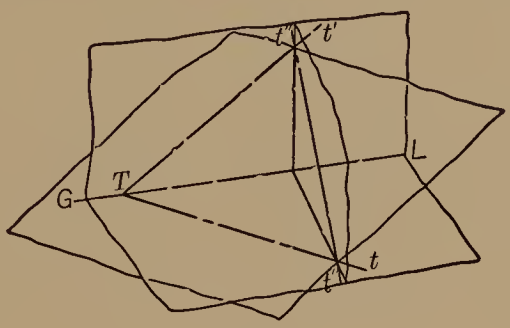


FIG. 1.

trace, and the intersection of T with P is called the P trace. In Fig. 1, plane T is shown in perspective. Fig. 2 shows plane T as represented in orthographic projection, likewise the manner of lettering the respective traces and the conventional line used to represent a plane trace; Tt' is the V trace, Tt, the H trace, and t'' $t,''$ the P trace.

TRACES OF A PLANE INTERSECT ON GL

(35) Since plane T, Fig. 1, is oblique to GL it must intersect GL in some point; this point is common to V and T, hence lies on their line of intersection, Tt'; it is also

common to H and T, hence lies on their line of intersection, *Tt*. The point in which T intersects GL has been proved common to *Tt'* and *Tt*, hence *Tt'* and *Tt* must intersect in a point on GL. Fig. 2 shows the traces as they appear when the coordinate planes have been revolved.

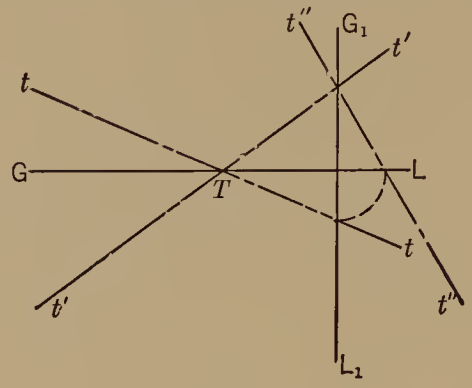


FIG. 2.

The H and V traces of a plane intersect in the point in which GL pierces that plane. Therefore, if a plane is parallel to GL this piercing point is at infinity, i.e., the H and V traces intersect at infinity and for our purposes are parallel to GL.

LOCATION OF A PLANE

(36) A plane is definitely determined by any two of its traces, for through these traces (two intersecting lines) one and only one plane can be passed.

THE PROFILE TRACE

(37) Planes are ordinarily given in terms of their H and V traces. This is because we solve the normal problem through the medium of the H and V planes. A plane can just as easily be given in terms of its V and P, or H and P traces, and sometimes is so given if any advantage can be derived from this form. The Profile trace is the intersection of the plane with P. This trace may also be defined

as the line passing through the points in which the H and V traces pierce the Profile plane. Since the Profile plane is revolved into coincidence with V about its intersection G_1L_1, with V, the Profile trace has always the direct relation with the V trace, Fig. 2, of intersecting it in the point in which the V trace intersects G_1L_1, since this point remains fixed in the revolution of the P plane. Since both H and P are revolved into coincidence with V about different lines in V, the P projection of the P piercing point of the H trace is revolved into coincidence with GL, while the H projection of this same piercing point is revolved into coincidence with G_1L_1. The P trace would have the same direct relation with the H trace that it normally has with the V trace, if it were our custom to revolve the P plane into coincidence with H about its intersection with H. This is but rarely done.

ALPHABET OF A PLANE

(38) The seven different characteristic positions in which planes may be conceived according to their relation to the planes of projection are named the **alphabet** of a **plane.** These positions are shown both in perspective and in orthographic projections in Fig. 3. Their relations to the planes of projection together with the characteristics of their traces are as follows:

1 Perpendicular to both H and V—Both traces perpendicular to GL.

2 Perpendicular to H and inclined to V { V trace perpendicular to GL. / H trace inclined to GL.

3 Perpendicular to V and inclined to H { V trace inclined to GL. / H trace perpendicular to GL.

4 Parallel to V—H trace parallel to GL, V trace at infinity.

5 Parallel to H—V trace parallel to GL, H trace at infinity.

6 Parallel to GL and inclined to H and V—Both traces parallel to GL.

7 Inclined to H, V, and GL—Both traces inclined to GL.

EXTENT OF PLANES

(39) The planes shown in Fig. 3 are not limited to any quadrant but pass through two, three, or four quadrants,

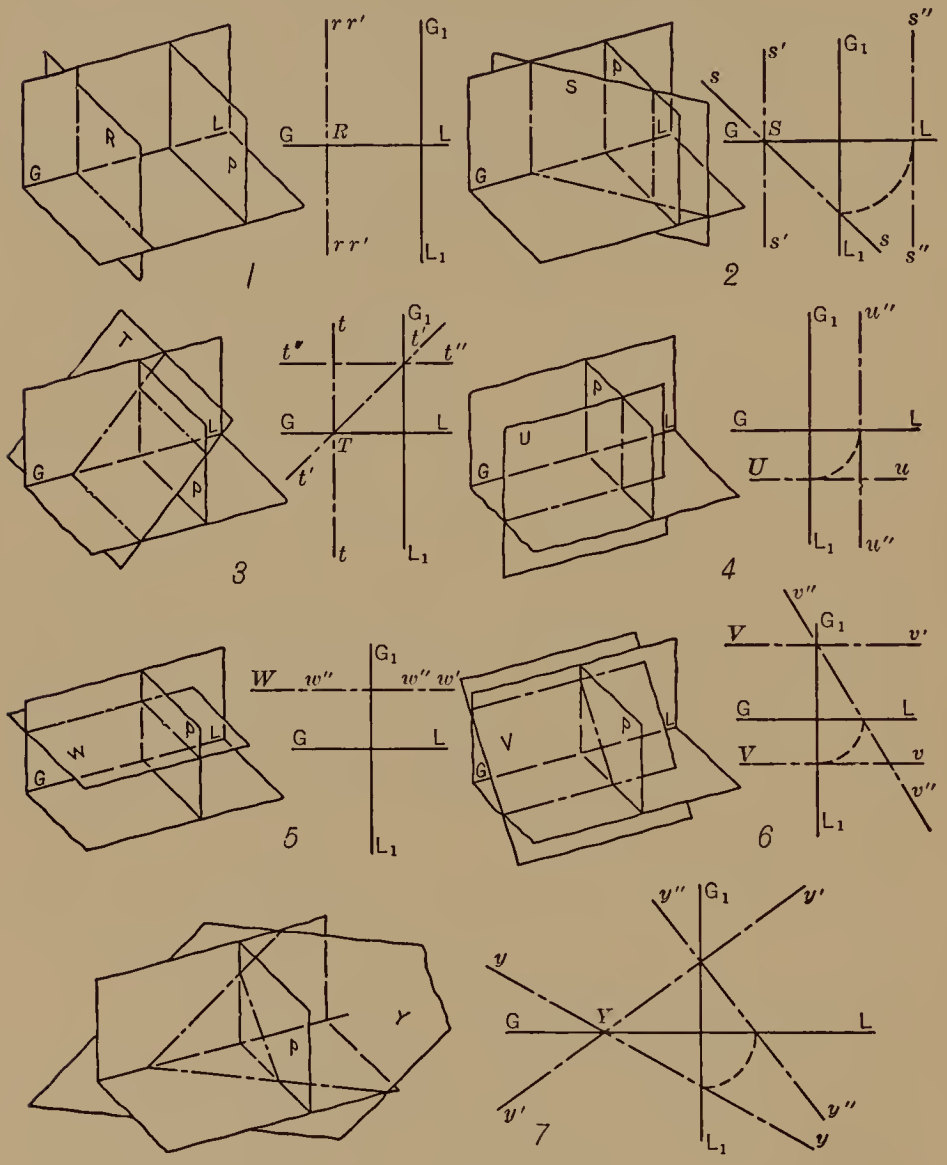


Fig. 3.

according to their position in any particular problem. Though the construction of a problem may deal with only

that part of a plane that lies in one quadrant, it must be understood that the plane is unlimited in extent. The traces therefore are not limited by GL. Fig. 4 represents four ways of representing the same plane.

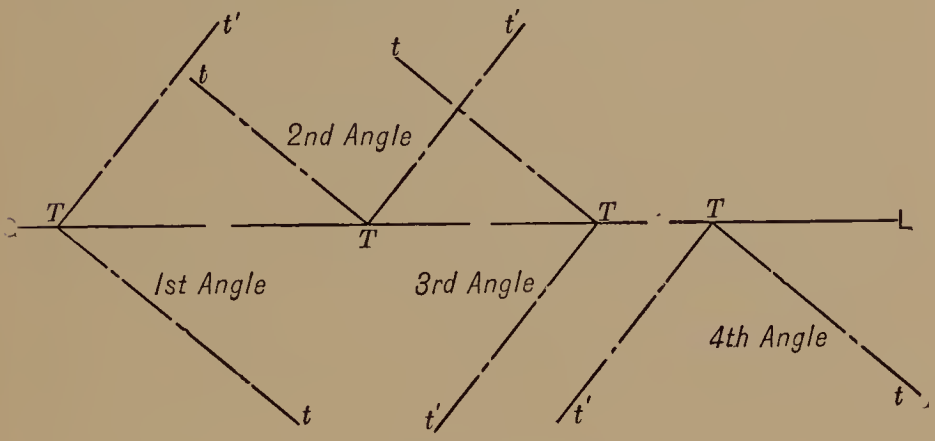


FIG. 4.

TO ASSUME A PLANE

(40) A plane whose position is not specified may be assumed at will and the traces drawn at will provided they intersect on GL, or GL produced to infinity if necessary; see Fig. 3.

PARALLEL PLANES

(41) If two planes are parallel their respective traces are parallel. For, if two parallel planes (the given planes), are intersected by a third plane (the V plane), the lines of intersection (the V traces) are parallel.

PERPENDICULAR PLANES

(42) The traces of perpendicular planes have not necessarily any definite relation to each other unless one of the planes is perpendicular to a plane of projection; in this case the traces in that plane are perpendicular to each other.

LINES PARALLEL TO PLANES

(43) The only requirement that a line need fill in order that it be parallel to a plane is that it be parallel to one line

of that plane. Since the lines of a plane may have all of the directions of the radii of a circle, the fact that a line is parallel to a plane implies nothing with respect to the relation of the directions of the projections of the line to the traces of the plane. One of the lines which may be passed through a point in space parallel to a plane is horizontal and its H projection will be parallel to the H trace of the plane. A second line lies in a plane parallel to V, and its V projection will be parallel to the V trace. A third line lies in a plane parallel to P, and its P projection will be parallel to the P trace, for, since the line is parallel to both the given plane and the plane of projection, it is parallel to their line of intersection. These three lines are the only lines parallel to the plane whose projections bear any definite relation in direction to the traces of the plane.

LINES PERPENDICULAR TO PLANES

(44) The given planes M and H intersect in the line M*m*, which may be considered the trace of plane M in plane H. Line AB is given perpendicular to plane M. Plane X through AB is the H projecting plane of AB, hence is perpendicular to H. The H trace of plane X is X*x*, which coincides with *ab*, the H projection of the line AB.

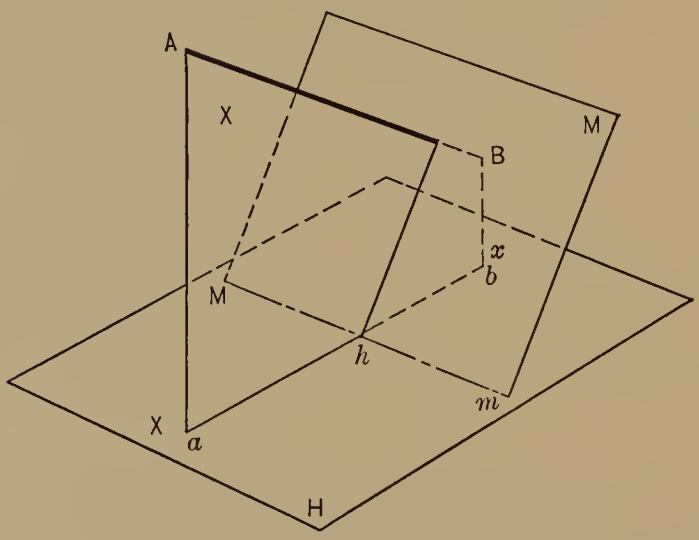


Fig. 5.

Plane X is perpendicular to plane M since it passes through the line AB which, by hypothesis, is perpendicular to the plane. Plane X is constructed perpendicular to plane H. It is therefore perpendicular to the line of intersection M*m* of these two planes, Art. 9*h*. Conversely, M*m* is perpendicular to plane X. It is therefore perpendicular to every line in plane X passing through its piercing point *h* in X; *ab* is one such line; *ab* is likewise the H projection of the line AB. The conclusion is then that: *the projections of a line which is perpendicular to a plane are perpendicular to the corresponding traces of that plane.*

CHAPTER IV

SIMPLE RELATIONS OF THE POINT, LINE, AND PLANE

(45) Problem 5.—**To find the H, V, and P piercing points of a given line.**

Solution:

(For the H and V piercing points.)

Since the H piercing point of a line lies in H, its V projection lies in GL; since it is a point on the given line, its projections lie on the corresponding projections of the given line. Similarly, the V piercing point is that point of the line which has its H projection on GL.

Construction:

Given: Line AB, Fig. 1.

Reqd: H and V piercing points of AB.

The intersection of $a'b'$ produced with GL gives h', the V projection of the H piercing point. A perpendicular to GL at h' intersects ab in h, the H projection of the H piercing point. ab produced, intersects GL at v, the H projection of the V piercing point; the intersection of a perpendicular to GL at v with $a'b'$, locates v', the V projection of the V piercing point.

Discussion:

Since the P piercing point of a line is a point in the P plane its H and V projections lie on G_1L_1.

Solution:

(For the P piercing point.)

The V projection p′, of the P piercing point is common to the

V projection of the line and G_1L_1. The H projection p, is common to the H projection of the line and G_1L_1.

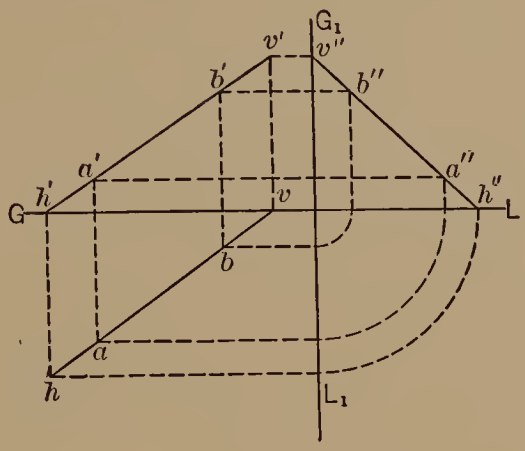


Fig. 1.

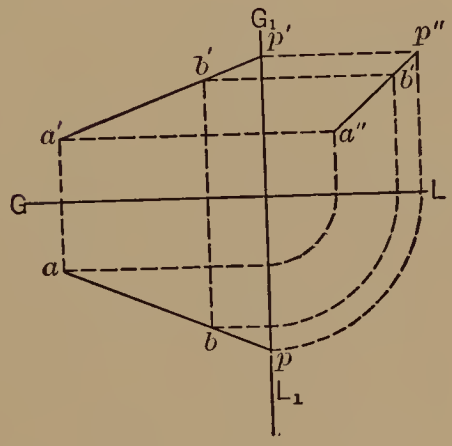


Fig. 2.

Whenever in any problem, it is required to find the Profile piercing point of a line, all three projections of that piercing point are usually shown. See Fig. 2.

The H and V piercing points of lines whose projections are perpendicular to GL, may be determined by referring the line to the Profile plane. The intersections of the Profile projection with GL and G_1L_1, are the Profile projections of the H and V piercing points. The projections of the H and V piercing points of every right line used in

Descriptive Geometry are lettered as in Figs. 1 and 2, hh', vv', and pp'.

(46) Problem 6.—**To assume a line in a given plane.**

If a line, inclined to H and V, lies in a plane T, Fig. 3, it pierces the V plane in a point common to both

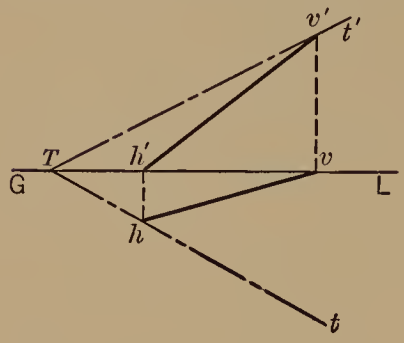


Fig. 3.

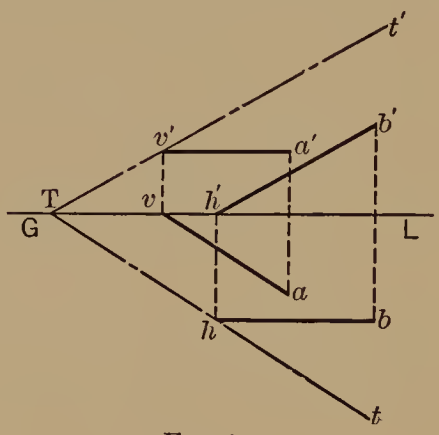


Fig. 4.

planes, therefore in a point on their line of intersection, the V trace. For similar reasons it pierces H in a point on the H trace. If a line, parallel to H or V, lies in a plane T, Fig. 4, it pierces one coordinate plane at infinity.

Any line which lies in a given plane, and is *parallel* to H, is called a **horizontal** of that plane, e.g., VA, Fig. 4.

Any line which lies in a plane T, and is *parallel* to V, is called a **vertical** of that plane, e.g., HB, Fig. 4.

Solution:

If any point on the V trace be assumed as the V piercing point, and any point on the H trace as the H piercing point of the required line, the line joining these two points is the required line. In the case of a Horizontal or Vertical of a plane, one of these piercing points will be at an infinite distance away on one trace of the plane. The projection will be parallel to that trace.

A horizontal of a plane then may be assumed by assuming its V piercing point. The V projection passes through v' *parallel* to GL and the H projection through v *parallel* to

the H trace. Any letter may be chosen to designate the other extremity. The method of assuming a vertical is similar to that of assuming a horizontal.

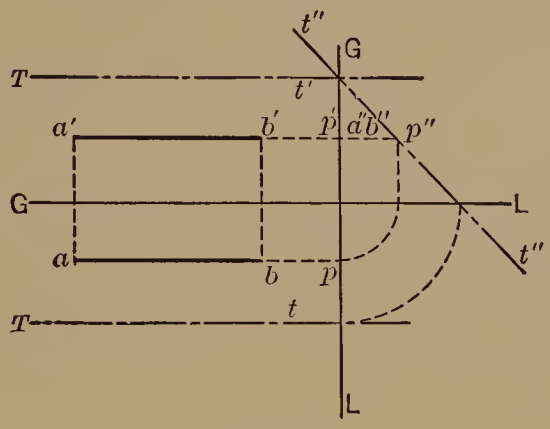


Fig. 5.

Obviously, a *horizontal* or a *vertical* is the easiest and simplest line that can be assumed in a plane, and its value as a working tool cannot be overestimated.

If the plane be parallel to the ground line the Horizontal or Vertical will also be parallel to GL. Reference to the profile plane, Fig. 5, shows that no difficulty is encountered in the construction of such a problem, as no new principle is involved, save that the piercing point assumed is the P piercing point.

(47) Problem 6: Example 1.—**To assume a point in a a given plane.**

Solution:

If a right line be first assumed in a given plane, any point of this line then will lie in the plane. If points on either trace of the plane suffice these may be assumed at once without the auxiliary line.

(48) Problem 6: Example 2.—**To pass a plane through a given line.**

Since by Art. 46, a line which lies in a plane T, pierces V and H in points which lie on the V and H traces

of T, obviously the converse of this proposition is true.

Solution:

If a plane passes through or contains a given line, its traces must pass through the respective piercing points of the line. These traces must intersect on GL, unless parallel to it.

(49) Problem 7.—**To pass a plane through two intersecting lines.**

Solution: (*a*)

Since the plane contains the two given intersecting lines, its traces pass through or contain the respective piercing points of these lines. These traces must intersect on GL unless parallel to GL.

Construction:

Given: Intersecting lines AB and CD, Fig. 6.

Reqd: Traces of the plane of AB and CD.

By Art. 45, the V projections of the V piercing points of AB and CD are v' and v', and the H projections of the H

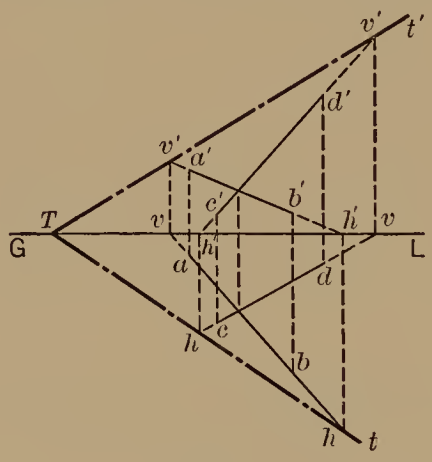


Fig. 6.

piercing points are h and h. The V trace of the required plane T passes through v' and v' and the H trace through h and h. The traces must intersect on GL, thus giving a check on the construction.

Discussion: (*b*)

Frequently one or more of the desired piercing points are inaccessible, or perhaps all may coincide in a point on GL. If then a third line be drawn from some point of one given line to a point of the second given line, this auxiliary line lies in the plane of the given lines.

Solution: (*b*)

If any convenient point of the one given line be connected with any point of the second given line, the piercing points of this auxiliary line are points of the traces of the required plane.

(50) Problem 7: Example 1.—**To pass a plane through two parallel lines.**

Solution:

Two lines cannot be parallel unless they lie in a common plane. The traces of this plane pass through the respective piercing points of the given lines.

(51) Problem 7: Example 2.—**To pass a plane through a given point and given right line.**

Solution:

A line through the given point parallel to or intersecting the given line converts this example into Problem 2 or Example 1. The plane of these lines is the required plane.

(52) Problem 7: Example 3.—**To pass a plane through three points.**

Solution:

The plane of the two intersecting or two parallel lines passing through the three given points is the required plane.

(53) Problem 7: Example 4—**Through one given line, to pass a plane parallel to a second given line.**

Solution:

If through any point of the first given line, an auxiliary line parallel to the second given line be passed, Art. 30, the plane of these two intersecting lines is the required plane.

(54) Problem 7: Example 5.—**Through a given point, to pass a plane parallel to a given line.**

Solution:

If through the given point a line parallel to the given line be passed, Art. 31, any plane passing through this line passes through the given point and is parallel to the given line.

(55) Problem 7: Example 6.—**Through a given point, to pass a plane parallel to two given lines.**

Solution:

If through the given point two lines parallel respectively to the two given lines be passed, the plane of these intersecting lines is the required plane.

(56) Problem 7: Example 7.—**Through a given point, to pass a plane parallel to a given plane.**

Discussion:

A plane through a given point and parallel to a given plane is the locus of all lines through the given point parallel to the given plane, and any two of these lines are sufficient to determine the plane.

If two planes are parallel their corresponding traces are parallel, Art. 41.

Solution:

Two lines passed through the given point parallel to the given plane will determine the required plane, their H and V piercing points giving two points on each of its traces.

Since the directions of the traces of the required plane are known, a single line through the given point parallel to the given plane will in general be sufficient for the solution.

Construction:

Given: Plane T and point O, Fig. 7.

Reqd: Plane through O parallel to T.

Through O, a line parallel to T is drawn; for the sake of simplicity a line parallel to a Horizontal of T is used. The V piercing point of this Horizontal is at V, and the V trace of the required plane S passes through v' parallel to Tt'; the H trace is then drawn parallel to Tt.

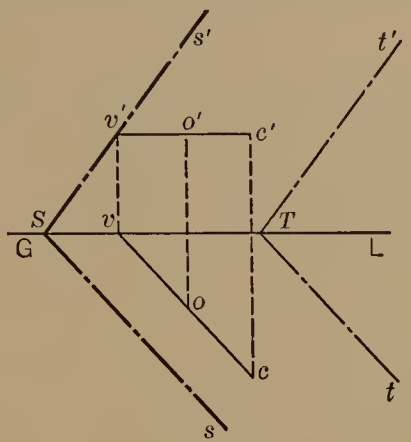


FIG. 7.

(57) Problem 6: Example 8.—**To pass a plane through a given point, perpendicular to a given line.**

Discussion:

The general method of passing a plane through a point

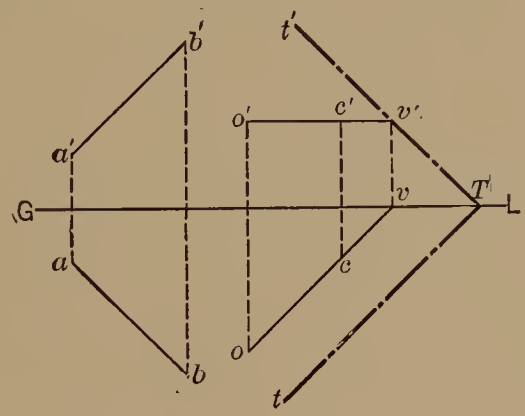


FIG. 8.

involves passing a line through the point and the plane through the line. From Art. 44 the direction of the traces of the required plane are known but not their position.

Since their direction is known a Horizontal or a Vertical of the required plane may be passed through the given point. The piercing point of this line is one point of the corresponding trace which is then determined since its direction is known.

Solution:

A horizontal or a vertical of the required plane through the given point has one piercing point through which one required trace passes perpendicular to the corresponding projection of the given line. This trace determines the second trace whose direction is known.

Construction:

Given: Line AB and point O, Fig. 8.

Reqd: Plane through O perpendicular to AB.

The Horizontal OC, is passed through O; its H projection, being parallel to the H trace of the required plane which is perpendicular to *ab*, is, therefore, perpendicular to *ab* and its V projection is parallel to GL. The V projection of its V piercing point is at v', through which the V trace is drawn perpendicular to $a'b'$. The H trace is then drawn perpendicular to *ab*.

(58) Problem 7: Example 9.—**To pass a plane through a given point perpendicular to a given plane.**

Discussion:

If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.

It should be noted that the corresponding traces of perpendicular planes are not in general perpendicular, but that when one of the planes, or both, is perpendicular to a plane of projection, their traces on that plane are perpendicular, Art. 42.

Solution:

If through the given point a line be passed perpendicular to the given plane, any plane passing through this line will be perpendicular to the given plane.

(59) Problem 7: Example 10.—**To pass a plane through a given line, perpendicular to a given plane.**

Solution:

If through any point of the given line a line be passed perpendicular to the given plane, the plane of this perpendicular and the given line will be the required plane.

(60) Problem 7: Example 11.—**To pass a plane through a given point perpendicular to two given planes.**

Solution: (*a*)

If through the given point two right lines be passed perpendicular respectively to the given planes, the plane of these two lines will be the required plane.

Solution: (*b*)

If through the given point a plane perpendicular to the line of intersection of the two given planes be passed, this plane will be perpendicular to both given planes.

(61) Problem 8.—**To find the line of intersection of two given intersecting planes.**

Discussion:

The line of intersection of two planes is determined by two of its points, or by one of its points and its direction. Since it is the line which is common to both planes, its piercing points are points on the corresponding traces of both planes; hence their points of intersection. These points, if available, are generally most convenient for determining the line.

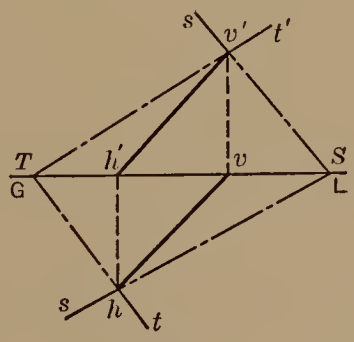


FIG. 9.

Solution:

The line which has for its H piercing point the intersection of the H traces, and for its V piercing point the intersection of the V traces, is the required line of intersection.

Construction:

Given: Planes S and T, Fig. 9.

Reqd: Line of intersection of S and T.

The intersection of the V traces of S and T is the V projection of the V piercing point of the line of intersection; its H projection, v, is on GL. The H projection, h, of the H piercing point of the required line, is the intersection of Ss and Tt; its V projection, h', is on GL; vh and $v'h'$ are the H and V projections of the required line.

SPECIAL CASES OF INTERSECTION OF PLANES

(62) 1. **If two corresponding traces are parallel** they may be considered as intersecting at infinity. The line of intersection then meets these parallel traces at infinity; i.e., it is parallel to them.

2. **If one plane is parallel to a coordinate plane,** e.g., to H, the line of intersection is parallel to H; i.e., it is a horizontal of both planes. It is therefore parallel to the H trace of the oblique plane, Art. 46.

3. **If both planes are perpendicular to H or V,** the line of intersection is perpendicular to H or V, Art. 24, 1 or 2; also Art. 9*h*.

4. **If both planes are parallel to GL,** neither the H nor V piercing point is available for determining the line. The line is parallel to GL, however, hence its position may be determined by reference to the profile plane or by the method of Case 5. The intersection of the profile traces of the planes is the profile piercing point of the required line.

5. **If either pair of traces do not intersect within limits of drawing, or all traces meet in the same point on GL,** two methods of solution are applicable.

Solution: (*a*)

An auxiliary plane, parallel to H or V, intersects the given planes in lines which intersect in a point of the required line of intersection. This point, with the available piercing point, determines the required line.

Construction:

Given: Planes S and T, Fig. 10 *a*.

Reqd: Line of intersection.

The auxiliary plane R, for convenience, parallel to V, intersects S and T, in right lines whose H projections, *ha* and *hb*, coincide with *Rr*, and whose V projections, *h'a'* and *h'b'*, are parallel respectively to the V traces of S and

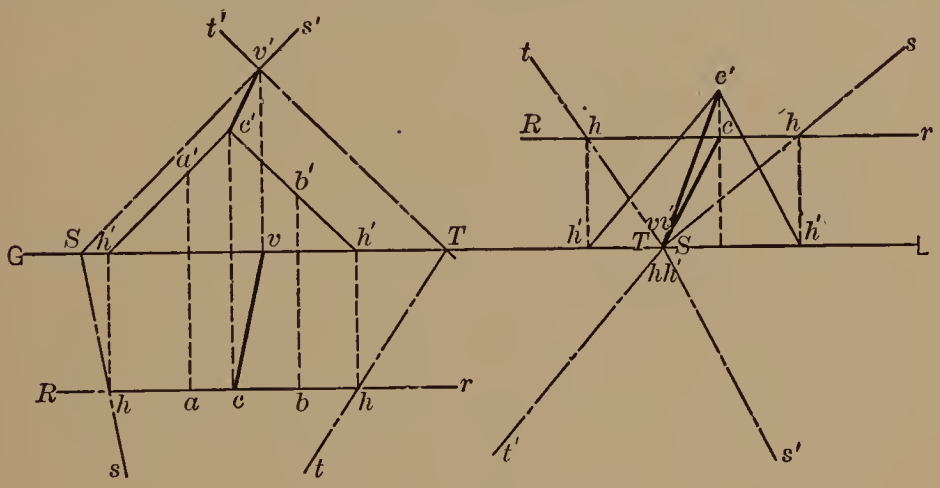


Fig. 10 *a*.

T, case 2. These lines intersect at C, a point on the required line. The line VC is the required line of intersection.

If neither pair of traces intersect within the limits of the drawing, two planes parallel to V (or H), may be employed, to secure two points on the required line.

Solution: (*b*)

If in any convenient location an auxiliary plane parallel to one of the given planes be constructed, its line of intersection with the other given plane is parallel to the required line of intersection. The required line passes through the point on GL in which the given traces intersect.

Construction:

Given: Planes S and T, Fig. 10 *b*.

Reqd: Line of intersection.

The auxiliary plane R is drawn parallel to T; $v'h'$ and vh are the projections of the line of intersection of R and S;

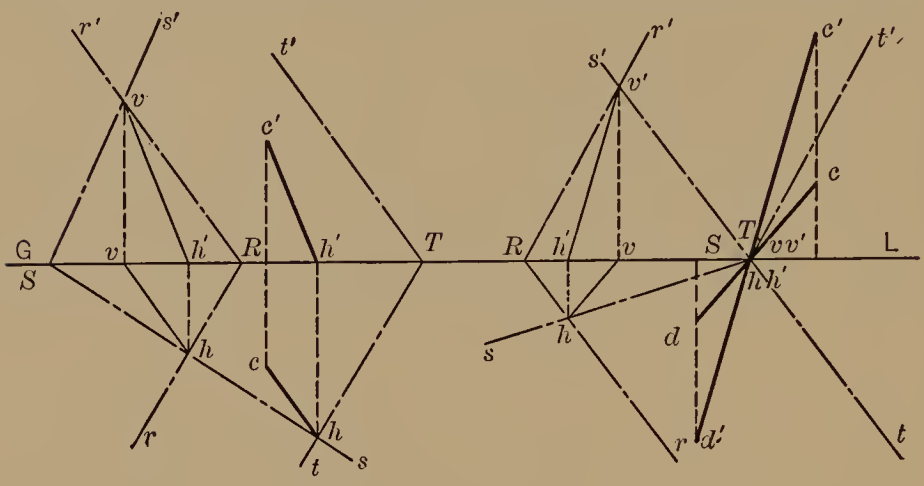


Fig. 10 *b*.

$h'c'$ and hc, the projections of the required line of intersection, are drawn parallel to $h'v'$ and hv.

(63) Problem 9.—**To find the point in which a given right line pierces a given plane.**

Solution:

If through the given line an auxiliary plane be passed (preferably either the H or V projecting plane of the given line), the required piercing point is a point of the line of intersection of this plane and the given plane, hence the point of intersection of the given line and the line of intersection.

Construction:

Given: Line AB and plane T, Fig. 11.

Reqd: Piercing point of AB in T.

An auxiliary H projecting plane, S, is passed through AB. The projections of the line of intersection of S and T are $h'v'$ and hv. The V projection of AB intersects $h'v'$ at p', the V projection of the required piercing point. The H projection of the piercing point P is at p on ab or hv.

(64) Problem 10.—**To find the projection of a given line on a given plane.**

Solution:

If through any two points of the given line (preferably the given points), perpendiculars to the given plane be passed, the line connecting the piercing points of these perpendiculars in the given plane, is the required projection.

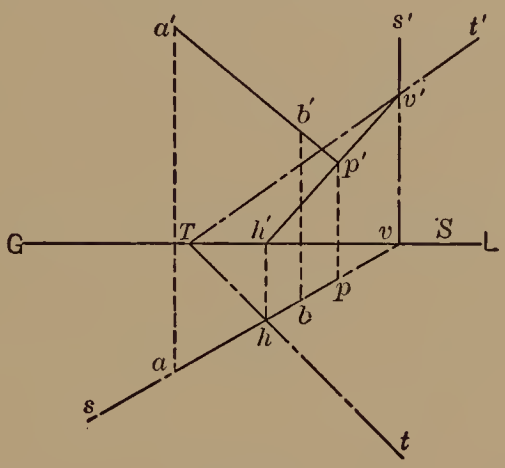


FIG. 11.

Construction:

Given: Plane T and line AB, Fig. 12.

Reqd: Projection of AB on T.

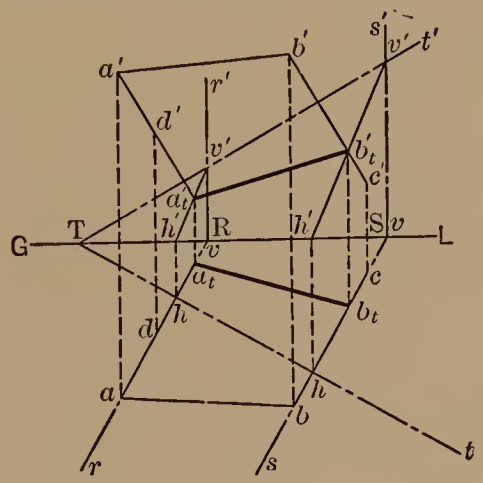


FIG. 12.

Perpendiculars to plane T are drawn from points A and B, Art. 44. The projections of the piercing points in T are $a_t a'_t$ and $b_t b'_t$, Art. 63; the projection of AB on T is then the line whose projections are $a'_t b'_t$ and $a_t b_t$.

SUMMARY OF RELATIONS OF LINE AND PLANE

(65) It may be well to summarize here certain relations concerning lines and planes, some of which may be determined by inspection of the given projections and traces, others of which cannot. These relations must be understood to take into account all three coordinate planes.

(1) If one line is parallel to another, the corresponding projections are parallel. Art. 31.

(2) If one plane is parallel to another, the corresponding traces are parallel. Art 41.

(3) If a line is perpendicular to a plane, its projections are perpendicular to the corresponding traces of the plane. Art. 44.

(4) If a line is parallel to a plane, its projections are **not** parallel to the corresponding traces of the plane, save in exceptional cases. Art. 43.

(5) If one line is perpendicular to another, their corresponding projections are **not** perpendicular, save in exceptional cases. Art. 33.

(6) If one plane is perpendicular to another, the corresponding traces are **not** perpendicular, save in exceptional cases. Art. 58.

CHAPTER V

HIGHER RELATIONS OF THE POINT, LINE, AND PLANE

REVOLUTION

(66) Problem 11.—**To revolve a given point about a given line, and to find its projections in any revolved position.**

Discussion:

The term " revolution " may be broadly defined as implying " motion in a circular path." Thus, if a given point, O, moves so that it remains in a given plane, T, and at a fixed distance from another given point, C, in that plane, point O will describe the circumference of a circle lying in plane T with center at C and radius CO; and point O is said to revolve about point C as a center. Or, if a line, AB, be taken perpendicular to plane T through point C, point O is said to revolve about line AB as an axis. Point O is termed *the revolving point;* plane T, *the plane of revolution;* point C, *the center of revolution;* line CO, *the radius of revolution;* line AB, *the axis of revolution;* and the circle described is called *the circle of revolution.*

It follows from the above that the problem of revolving a given point about a given axis involves only the determination of a certain definitely fixed circle. It will be recalled from plane geometry that the determination of a circle involves the fixing of the following three characteristics of the circle: its plane, its center, and its radius. When these three elements are fixed, the circle, and every point on its circumference, is determined. Since the plane of revolution contains the revolving point and is perpendicular to the axis of revolution, the plane is determined,

and knowing that the center of revolution lies on the axis of revolution and in the plane of revolution, the center is determined. Since the revolving point is one point on the circumference of the circle of revolution the radius of the circle of revolution is determined.

Applications of Revolution:

(67) The problem of revolving a point about a straight line finds many applications in the further problems of Descriptive Geometry, especially when extended to the revolution of lines and planes. For example, since a line is shown in its true length by its projection on a plane only when the line is parallel to or lying in that plane, a suitable axis may be chosen and the line revolved into a position parallel to or into coincidence with one of the planes of projection, in which position its true length can be measured. As another example, any plane which is not already parallel to a plane of projection may be revolved into a position parallel to or coincident with one of the planes of projection. Thus, the angle between two intersecting lines, or the distance between parallel lines may be measured, or the true size and shape of any plane figure lying in that plane determined.

(68) Problem 11: Example 1.—**To revolve a given point O about a line AB, which is perpendicular to H, and find the points in which its path pierces V.**

Solution:

If a plane be passed through point O perpendicular to the given line AB, it will intersect the line in a point which will be the center of the circular path in which O travels. The distance from O to this point will be the radius of the circle. The circular path will pierce the V plane in points of the V trace of its plane and at a distance from its center equal to the radius of revolution.

Construction:

Given: Point O and line AB, Fig. 1.

Reqd: The V piercing points of the path of O, as it revolves about AB as an axis.

Tt' is the V trace of the plane through O perpendicular to AB. Plane T intersects line AB at point C, which is the

center of the path of O. The radius of the circle is *oc*, and *c* is the center of the H projection of the circle, which in this

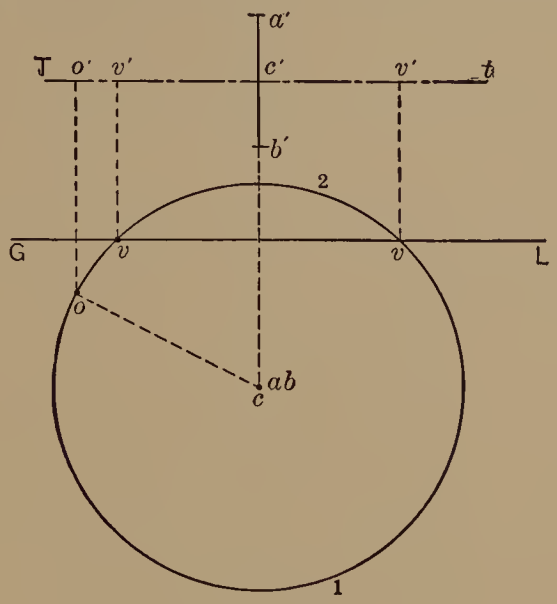


FIG. 1.

case is identical in size and shape with the circle itself. The H projection of the circle intersects GL in points *v* and *v*, the H projections of the V piercing points of the path of O. The V projections of these piercing points are *v'* and *v'*, on the V trace of the plane.

(69) Problem 11: Example 2.—**To revolve a given point O about a given line AB, which lies in the H plane, and find the points in which the path of O pierces H, V, and P.**

Solution:

In revolving about axis AB point O moves in a circular path which lies in plane T passed perpendicular to the axis AB through point O. The center of the path is point C in which AB pierces T. This circle will pierce H, V, and P at the points in which it intersects the corresponding traces of T.

Discussion:

Since the axis of revolution lies in H, the plane of revolution, being perpendicular to the axis, will be per-

pendicular to H. All points on the axis being in H, the point in which the axis pierces the plane of revolution will be a point in H, on the H trace of the plane. With its center in H, one-half of the circle will lie above and the other half below H, with one diameter in the H trace of the plane, Fig. 2.

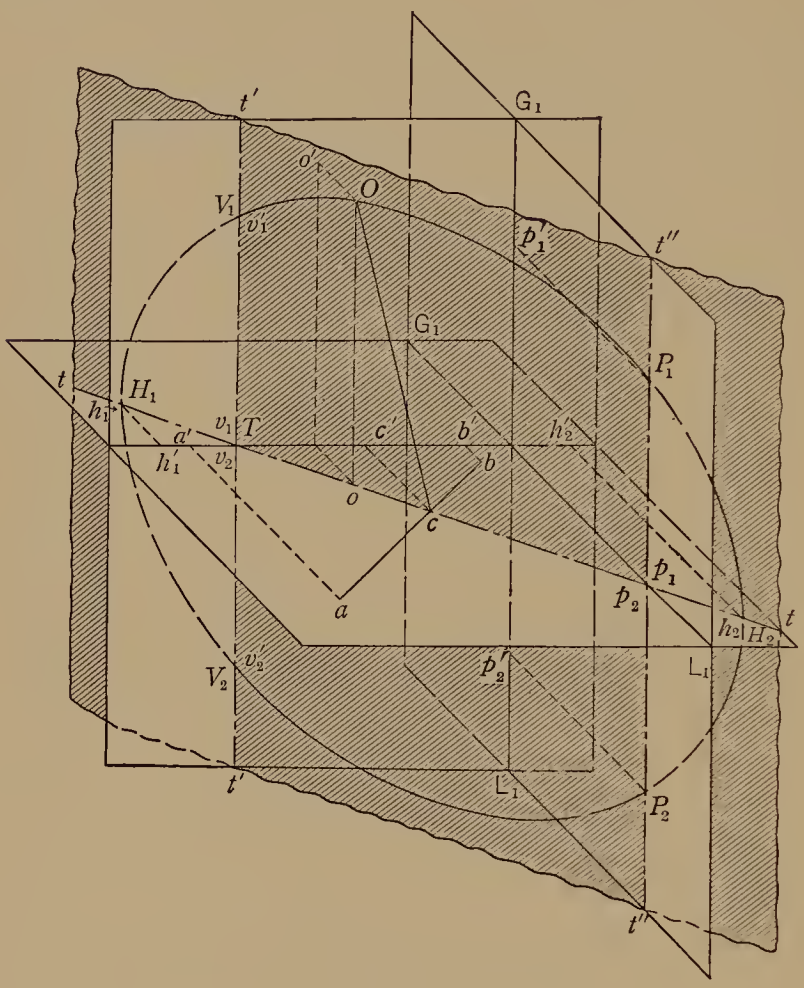


Fig. 2.

The plane of revolution being thus an H projecting plane, all points on the circle will have their H projections in its H trace. As the given point moves around the circle its H projection moves along the H trace, always perpendicularly below or above the moving point; i.e., the H projection of the circle is a straight line equal to a diameter

and lying in the H trace of the plane. The V projection of the circle will in general be an ellipse, but it will not be necessary to construct this ellipse for the solution of any problem. Instead, the plane of revolution will be revolved about its H trace into H, in which position the circle will show in its true size and shape in the H projection, Fig. 3. The H projecting line of any point on the circle will, in this revolved position, fall perpendicular to the H trace at the H projection of the point. The length of this perpendicular,

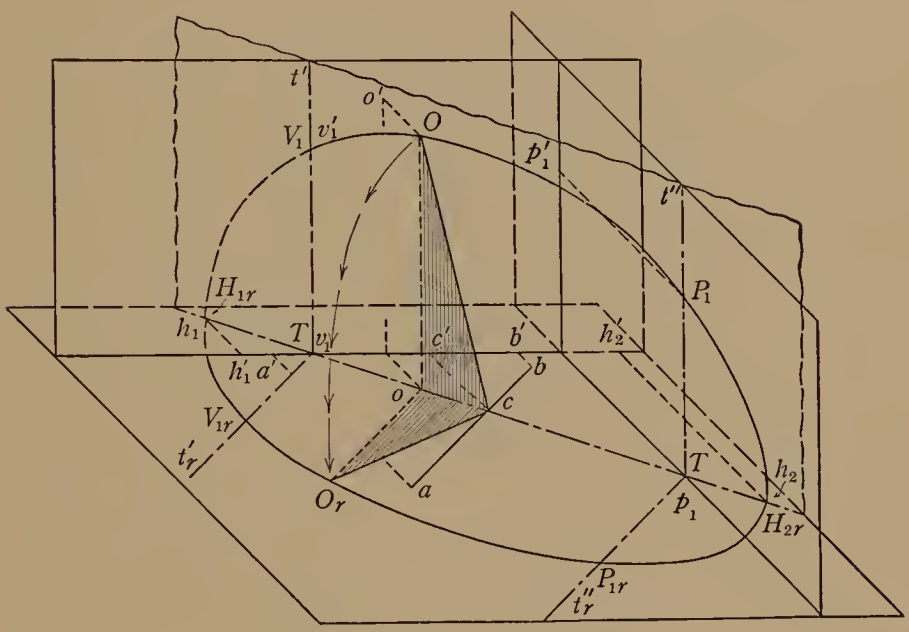


FIG. 3.

being the distance of the point above or below H on the original position of the circle, is equal to the distance of the V projection from GL. If the position of the axis and length of the radius are such that the circle in its original position pierces V and P, the revolved positions of these piercing points are found at the points in which the revolved circle intersects the revolved positions of the V and P traces. Since the plane is revolved about its H trace into H, this trace and the points in which it is intersected by the circle remain in their original positions.

Since the V and P traces of the plane of revolution are perpendicular to its H trace at the points in which the H trace intersects GL and G_1L_1, respectively, they will fall perpendicular to the H trace at these points when the plane is revolved into H.

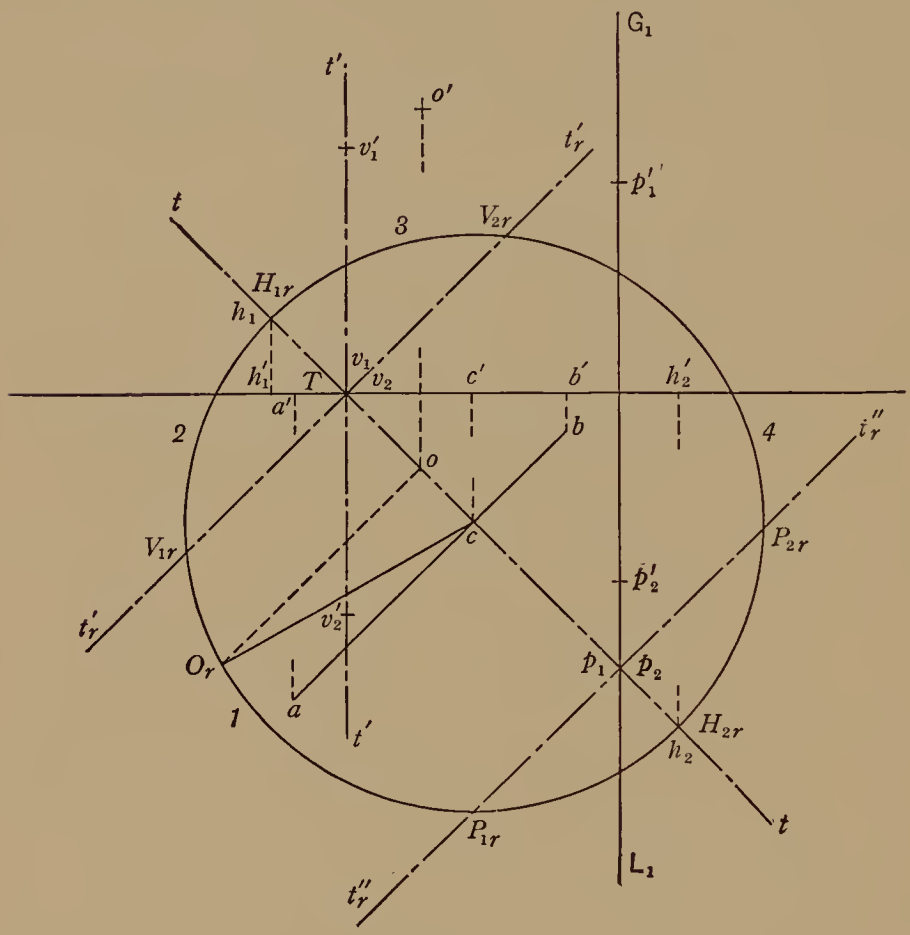


Fig. 3 *a*.

The plane in its original position passes through all four quadrants, the part in any quadrant being bounded by one of the four right angles formed by the intersection of the H and V traces. In the revolved position these four right angles are shown in their true size between the H trace and the revolved position of the V trace. The quadrant in which the given point lies is known and every other point

whose revolved position is in the same angle between the H and V traces must lie in that same quadrant. The quadrants which contained the other three right angles may be determined by comparing their positions relative to the H and V traces in the revolved position with that of the known quadrant in which the given point lies. Thus, if the given point is in the first quadrant, in the revolved positions all points on the same side of the V trace as the given point but on the opposite side of the H trace are in the fourth quadrant, etc.

Construction:

Given: Point O in the first quadrant, and line AB in the H plane of projection. (Figs. 2, 3, and 3 *a*.)

Reqd: The H, V, and P piercing points of the circular path of O as it revolves about AB as an axis.

Plane T, passed through O perpendicular to AB, being an H projecting plane, has its H trace through the H projection of O perpendicular to the H projection of AB. Its V trace is perpendicular to GL and also perpendicular to tTt. In space the P trace is perpendicular to tTt at the point in which the latter intersects G_1L_1, but since the solution is to be made in H and V projections, the usual P projection of the P trace will not be shown.

By inspection, the center of revolution is on tTt at its intersection with AB.

Having determined the plane and center of revolution, the plane is revolved about its H trace into the H plane. In this revolution the given point may be made to fall on either side of the H trace. Oo, the H projecting line of O, falls perpendicular to tTt through the H projection of O. Point O falls at O_r on this perpendicular at a distance from tTt equal to the distance of o' from GL. The radius of revolution is shown in its true length at cO_r and the circle is drawn in its true size and shape. The V and P traces fall at $t'_rTt'_r$ and $t''_rTt''_r$, perpendicular to tTt at the points in which tTt intersects GL and G_1L_1, and the revolved positions of the V and P piercing points of the circle are

at V_{1r}, V_{2r}, and P_{1r}, P_{2r}, where the circle intersects these traces in the revolved positions. Since the plane was revolved about its H trace, tTt and the points H_1 and H_2 in which the circle interests it, remain in their original positions.

Having determined any point, such as these piercing points, on the revolved position of the circle, the H projection is in tTt at the foot of the perpendicular from the point on the circle. The V projection is above or below GL at a distance equal to the length of this perpendicular. Thus the H projection of the V piercing point, V_1, is at v_1, in tTt at the foot of the perpendicular from V_{1r}, and v'_1 is above GL at a distance equal to $V_{1r}v_1$.

Points which on the original position of the circle are above H will all have their revolved positions on the same side of tTt as O_r, since O is given above H; points on the opposite side of tTt are below H. Likewise, points having their revolved positions on the same side of $t'_rTt'_r$ as O_r must be in front of V in their original positions; those on the opposite side are behind V. In this manner it may be determined that all points in each of the four right angles between tTt and $t'_rTt'_r$ were in the quadrants indicated by the numbers shown on the circumference of the revolved circle.

(70) Problem 11: Example 3.—**To revolve a given point about a given line in H or V, through a given angle "X", and find the projections of the point in this position.**

Solution:

The point revolves about the line in a circle whose radius equals the distance from the point to the given line, and whose plane is perpendicular to the given line. The angle "X" is the angle included between the radius through the given point and a second radius through a second or required point of the circle. The projections of this second point of the circle, or second and third points, since there may be two, are the projections of the required revolved position or positions of O.

Construction:

Given: Line AB and point O, Fig. 4.

Reqd: To revolve O about AB through 90°, into the 4th quadrant.

Plane T is passed through O, perpendicular to AB. Its H trace, tTt, passes through o, perpendicular to ab; the V trace is perpendicular to GL. After plane T is revolved about tTt into H, O is found in the position o_r, in the 1st quadrant on the revolved circle of revolution; o_r is then moved along the circumference of the circle in the direction of the 4th quadrant through an arc of 90°, to the required position at o_{4r}, the radii $o_{4r}c$ and o_rc making 90° with each other. As O moves along the circle, its H projection moves along the H trace of T, always perpendicularly below or above the moving point.

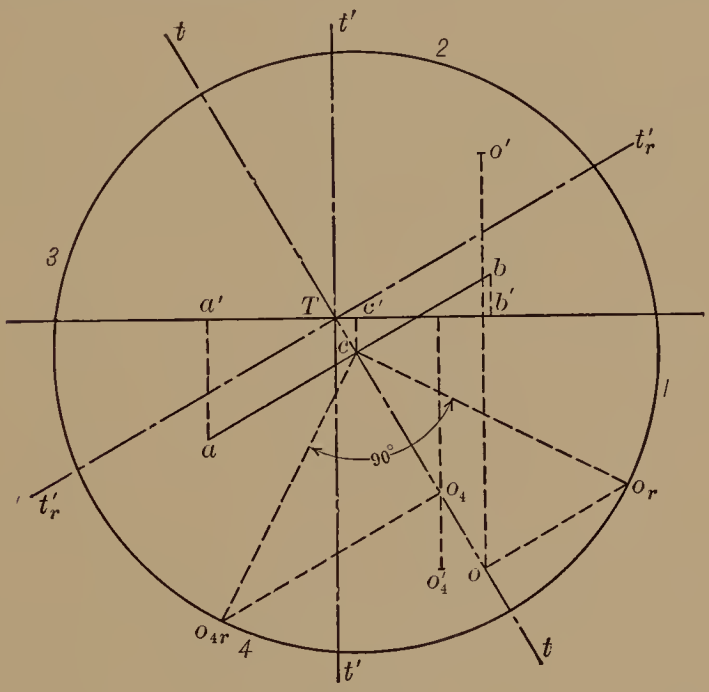


FIG. 4.

The H projection of o_{4r} is the intersection o_4 of the perpendicular from o_{4r} to Tt, with Tt. (The sub numeral should correspond to the quadrant in which the point is found, e.g., o_4 for 4th quadrant, etc.) The distance o_4o_{4r} represents the distance that O is below H; a projecting line is drawn through o_4, perpendicular to GL, and a distance measured off on it below GL equal to o_4o_{4r}; this

locates the V projection o'_4. The new or required position of the point is now determined by the two projections o_4 and o'_4.

(71) Problem 11: Example 4.—**To revolve a given point about a given line in H or V, into a position in a given quadrant at a given distance from H or V.**

Discussion: (*a*)

As a point is revolved about a line in H, the distances from the several positions of the moving point to H are equal to the distances from these positions of the point to the H trace of the plane of revolution.

Solution:

(Line in H and distance referred to H.)

The plane of the circle of revolution passes through the given point perpendicular to the given line. The distance from the point to the line is the radius of the circle. The distances of the moving point from H are equal to the vertical ordinates of the circle at the various positions considered.

Construction: (*a*)

Given: Line AB and point O, Fig. 5.

Reqd: To revolve O about AB into the 4th quadrant, $\frac{1}{2}''$ below H.

Plane T is passed through O perpendicular to AB; tTt passes through o perpendicular to ab, Tt' perpendicular to GL. After plane T is revolved about tTt into H, O is found in the position o_r, and the circle of revolution is constructed with radius o_rc. Since plane T has been revolved into H about its H trace, the distances from any points on the circumference of the circle to tTt represent the distances which the positions of these same points were from H before T was revolved; i.e., the distances the V projections of the points are above or below GL. To find the point of a circle, in the 4th quadrant, which is $\frac{1}{2}''$ below H, a perpendicular to tTt, $\frac{1}{2}''$ long, and *below* tTt is erected and through its extremity a line is drawn parallel to tTt; this line intersects the circle at o_{4r}, the required point. The H

projection of O_4 is o_4 and the V projection o'_4, below GL a distance equal to $o_{4r}o_4$.

Discussion: (*b*)

(Line in H but distance referred to V.)

As a point is revolved about a line in H, no ordinates nor distances in the circle indicate the distances which points are from the V plane. It is known, however, that as

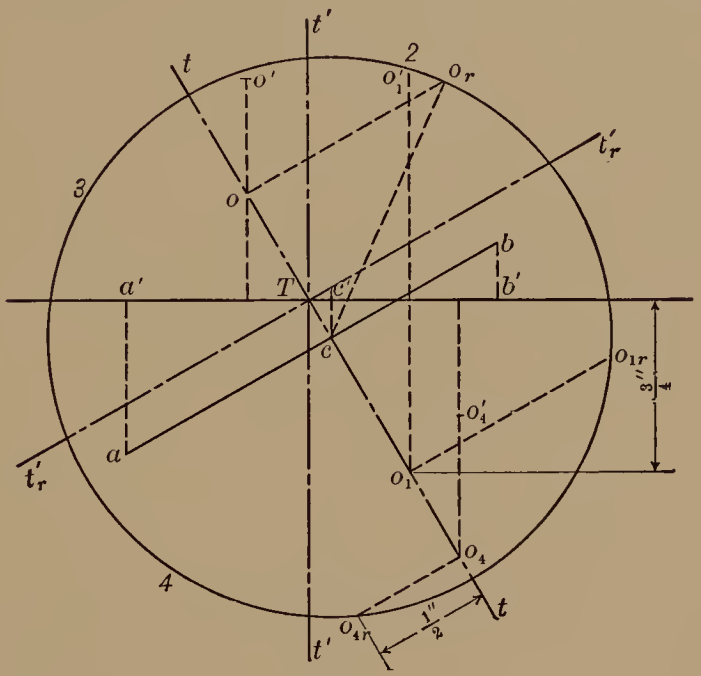


FIG. 5.

the point revolves about the line, its H projection lies always in the H trace of the plane of revolution, and the *distances from the* H *projections of the moving point to* GL *indicate the distances from the point to* V.

Construction: (*b*)

Given: Line AB and point O, Fig. 5.

Reqd: To revolve O about AB into the 1st quadrant, $\frac{3}{4}''$ before V.

Since the point is to be $\frac{3}{4}''$ before V, its H projection will be $\frac{3}{4}''$ *below* GL; since it is in plane T, its H projection

is on the H trace tTt. A perpendicular is then drawn to GL $\frac{3}{4}''$ long, and through its extremity a line is passed parallel to GL and extended until it intersects the H trace tTt at o_1, the H projection of the required position. A perpendicular to tTt at o_1 intersects the circle at o_{1r}, the revolved position of the required point; o_1o_{1r} is then the distance from this position of the point to H; the V projection, o'_1, is then found above GL a distance equal to o_1o_{1r} and on the perpendicular to GL through o_1.

(72) Problem 12.—**To find the true length of a given line.**

Solution: (*a*)

If the given line be revolved about its H or V projection, into the corresponding plane of projection, it will show in its true length in this revolved position.

Construction: (*a*)

Given: Line AB, Fig. 6.

Reqd: True length of AB.

Revolving AB about its V projection into V is equiv-

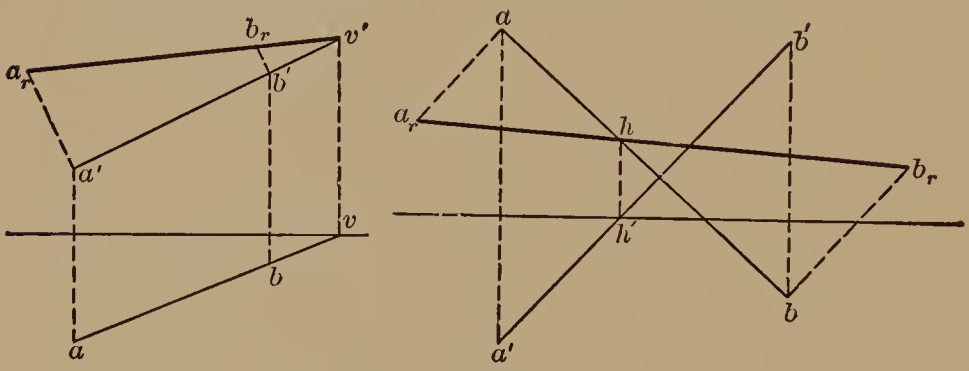


Fig. 6.

alent to revolving the V projecting plane of the line about its V trace into V, in the same manner as the plane T was revolved in the previous problems, the revolved position of AB being found as was that of O. We therefore erect perpendiculars to $a'b'$, at a' and b', equal in length to the distances from a and b to GL. The line a_rb_r, connecting their extremities is the required revolved line and *true length.*

The revolved position of the line, produced, must always pass through the piercing point of the given line. Fig. 6.

Solution: (*b*)

If the given line be revolved about the H projecting line of one extremity until it is parallel to V, its true length will then be shown on the V plane.

Construction: (*b*)

Given: Line AB, Fig. 7.

Reqd: True length of AB.

As AB revolves about the H projecting line of B, the H projection *ab* moves with *b* as a center into the position *ab*

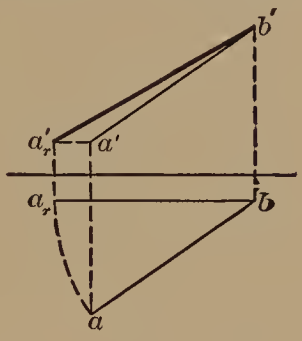


Fig. 7.

parallel to GL. A remains the same distance above H in this revolution, therefore its V projection *a'* moves from *a'* along a parallel to GL to *a'r*. *a'rb'* is the required true length.

(73) Problem 13.—**To find the distance from a given point to a given plane.**

Solution:

If through the given point a perpendicular to the given plane be passed, the true length of the portion of the perpendicular between the given point and the piercing point will be the required distance.

Construction:

Given: Plane T and point O, Fig. 8.

Reqd: Distance from O to T.

A perpendicular to plane T is drawn through O, Art. 44; p and p' are the projections of its piercing point in T, Art.

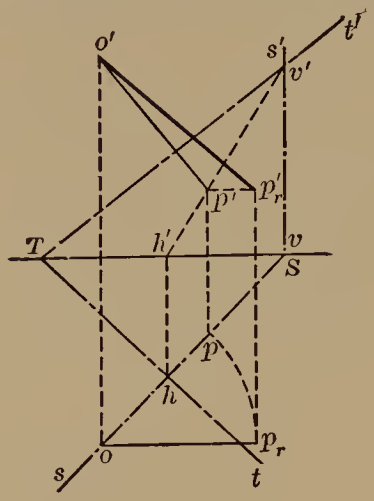


Fig. 8.

63; $o'p'_r$ is the true length of OP and the required distance, Art. 72.

(74) Problem 13: Example 1.—**To find the distance between two parallel planes.**

Solution:

If through any point in one plane S, a perpendicular to plane T be passed the true length of the portion of the perpendicular from

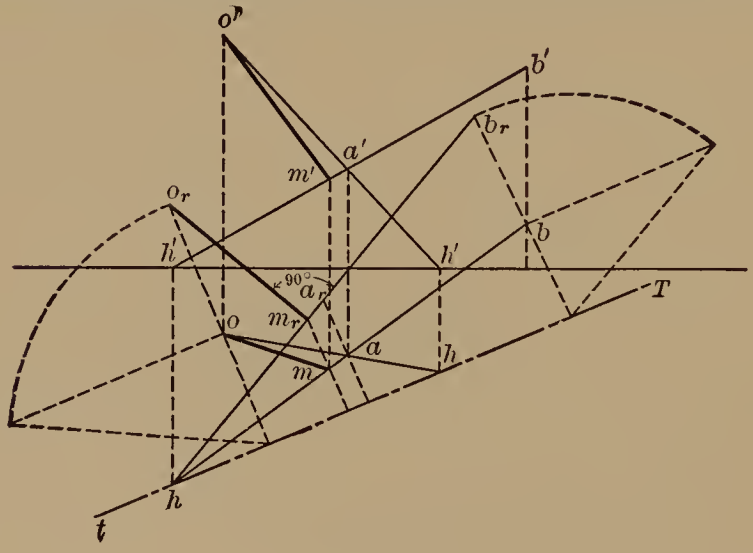


Fig. 9.

the assumed point in S to its piercing point in T will be the required distance.

(75) Problem 14.—**To find the projections and true length of the shortest line from a given point to a given line.**

Solution: (*a*)

If a plane be passed through the given point and line and revolved about either of its traces into the corresponding plane of

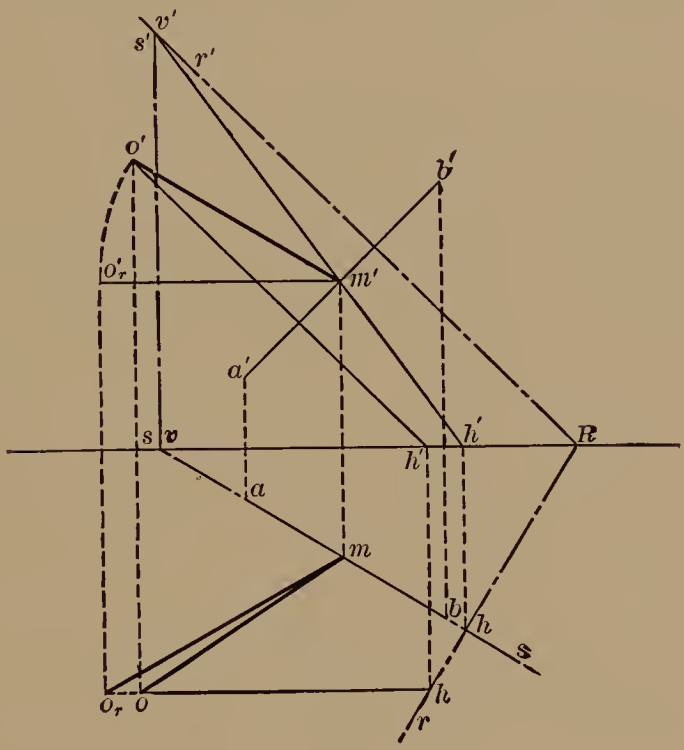


Fig. 10.

projection, the perpendicular from the revolved position of the point to the revolved position of the line will be the revolved position and true length of the required line.

Construction:

Given: Line AB and point O, Fig. 9.

Reqd: Distance from O to AB.

Plane T, of the given point and line, is determined by means of AB and the auxiliary line OA, Art. 51. Its H

trace is Tt. On revolving plane T about Tt into H, O is found at o_r and AB at a_rb_r; the perpendicular o_rm_r, from o_r to a_rb_r, is the required true distance. The projections of this perpendicular are om and $o'm'$; m being located by the intersection with ab of a perpendicular to Tt from m_r.

Solution: (*b*)

If a plane be passed through the given point perpendieular to the given line, the distance from the given point to the piereing point of the given line in this plane will be the required distanee, Fig. 10.

(76) Problem 14: Example 1.—**To find the true distance between two given parallel lines.**

Solution: (*a*)

If the plane of the two parallel lines be revolved about either of its traees into the corresponding plane of projection, the lines will show in their true relative positions. The perpendicular distance between the revolved positions of the parallel lines is the required distanee.

Solution: (*b*)

Solution (b) of Problem 14, Art. 75, is equally applicable to this problem.

The distance between two parallel lines is equal to the perpendicular measured from any point on one line to the other line.

(77) Problem 15.—**To find the projections and true length of the common perpendicular to two non-intersecting and non-parallel lines AB and CD.**

Solution:

If through the line AB a plane T be passed, parallel to CD, and the projeetion of CD on plane T be found, the perpendicular to plane T, at the point in whieh the projection of CD on T interseets AB, will interseet CD and is the required eommon perpendieular.

Construction:

Given: Lines AB and CD, Fig. 11.

Reqd: Common Perpendicular to AB and CD.

AE is passed through point A parallel to CD, Art. 31;

the plane of AB and AE, is the required plane through AB parallel to CD, Art. 53. Through C and D, the perpendiculars, CF and DG, to plane T are drawn; their piercing points are C_t and D_t, and the projection of CD on T is C_tD_t, Art. 64. From point O, at which this projection intersects AB, the required perpendicular to plane T is erected, Art. 44; it intersects CD at M. $o'm'$ and om are then the projections, and $o'm'_r$ the true length of the common perpendicular.

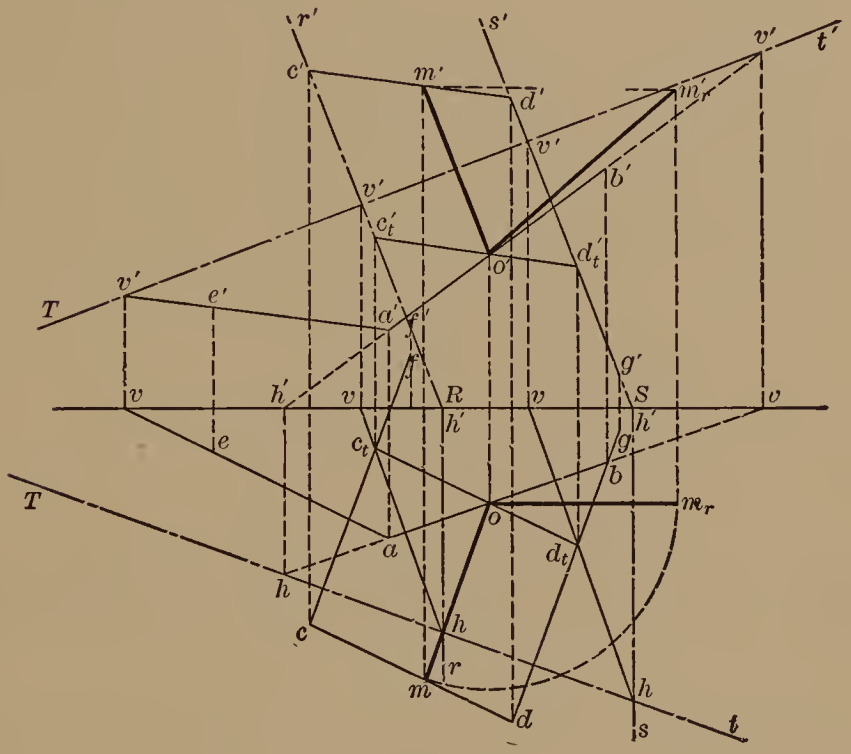


Fig. 11.

(78) Problem 16.—**To find the true value of the angle "X" between two intersecting lines.**

Solution:

If a plane be passed through the two intersecting lines and be revolved about either of its traces into the corresponding plane of projection, the relation of the lines to each other, *i.e.*, the angle between them, does not change in the revolution and the angle

between the revolved positions of the given lines is the true value of the required angle " X."

Construction:

Given: Lines AB and CD, Fig. 12.

Reqd: Angle " X."

R is the plane of AB and CD. Plane R is revolved about Rr' into V; point O falls at o_r; the piercing points V and V of AB and CD, being on the axis of revolution, are fixed; the lines AB and CD hence revolve into the positions $o_r v'$ and $o_r v'$; $v' o_r v'$ is then the true value of angle " X."

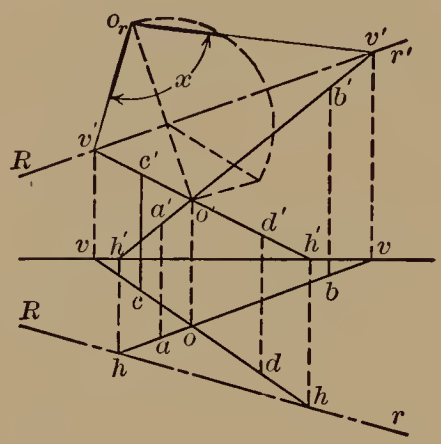


Fig. 12.

(79) Problem 16: Example 1.—**To find the projections of the bisector of the angle " X " between two intersecting lines.**

Discussion:

This problem illustrates the very general problem of finding the projections of a point, line, or figure, lying in a plane, when its revolved position is known. The construction is given in three forms, each of which is applicable under varying circumstances and all of which are equally important.

If the bisector is found to intersect the trace which is used as the axis of revolution, the point of intersection may be used most conveniently in counter-revolution. This is the basis of the first construction. If the bisector does not

intersect this trace within convenient limits, an auxiliary line may be taken in such a position as to intersect the two given lines and the bisector; this line is then used to secure the projections required, as in the second construction. The more general method is independent of any piercing point or auxiliary line, and is shown in the third construction. It is applicable to the greatest variety of cases. One fact may well be kept in mind, that the bisectors of the projections of an angle *are not necessarily* the projections of the bisector.

Solution:

If the plane of the given lines be revolved about either of its traces into the corresponding plane of projection, the angle " X " will show in its true value and may be bisected by a right line. If any point of this bisector be revolved back into its true position a right line through this point and the vertex will be the required bisector.

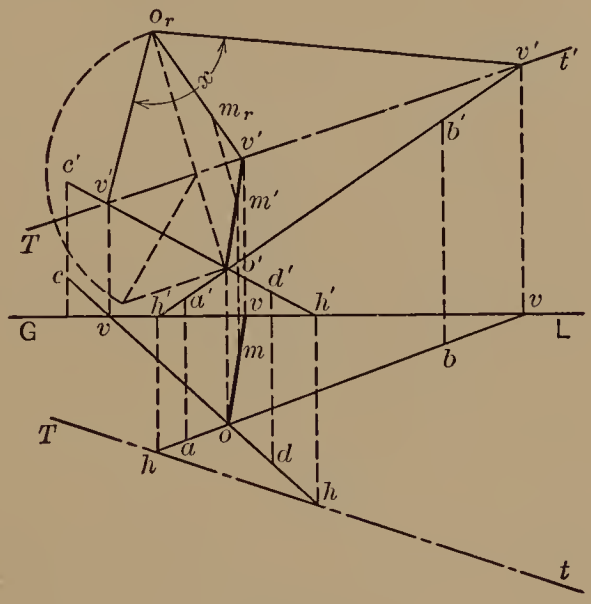


Fig. 13.

Construction: (*a*)

Given: Lines AB and CD, Fig. 13.

Reqd: Bisector of angle " X."

The traces of T, the plane of AB and CD, are Tt' and

Tt, Art. 49. Revolving plane T about *Tt'* into V, O falls at o_r, and $v'o_rv'$ is the true value of angle X. The bisector of angle $v'o_rv'$ is o_rm_r. This line is produced until it meets the V trace at *v'*. The line joining this point with *o'* is the vertical projection of the bisector. The horizontal projection is then obtained by connecting *o* with *v*.

Construction: (*b*)

Given: Lines AB and BC, Figs. 14 *a* and 14 *b*.

Reqd: Bisector of angle " X."

The traces of T, the plane of AB and BC, are *Tt'* and *Tt*. The plane is revolved about *Tt'* into V and a_rb_r, b_rc_r

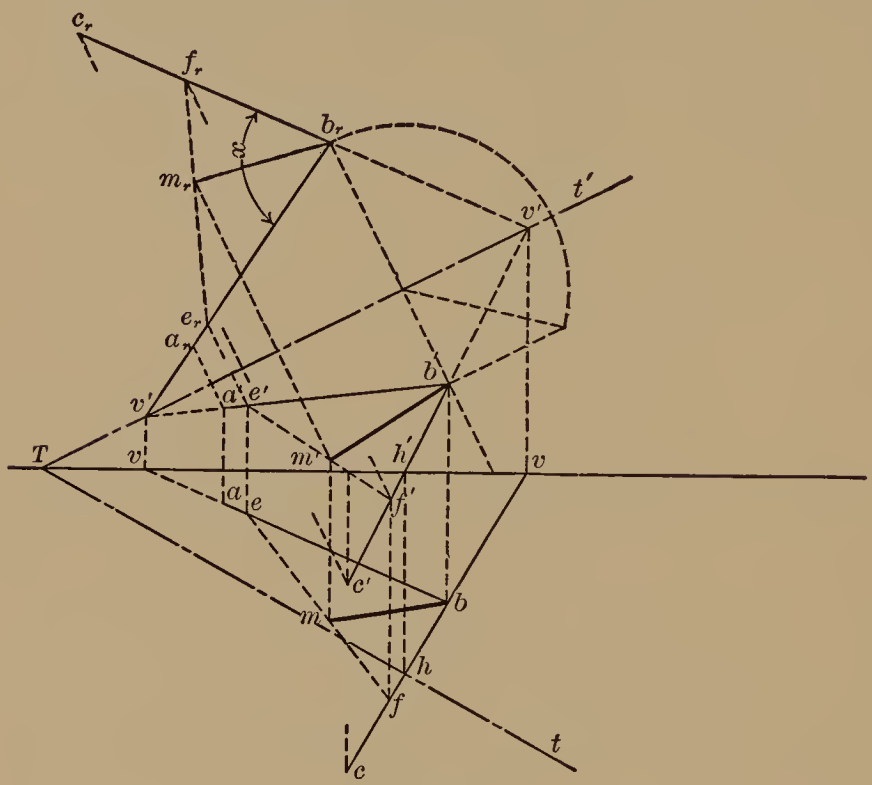


Fig. 14 *a*.

and b_rm_r are drawn as in the construction above. e_rf_r is drawn intersecting a_rb_r at e_r, b_rc_r at f_r and b_rm_r at m_r. Since e_r represents a point on AB, its V projection is found by passing through it a perpendicular to *Tt'* and extending

it until it intersects $a'b'$; its H projection is located by drawing through e' a perpendicular to GL till it intersects ab. The projections of F, which is a point on BC, are found in like manner. M is a point on EF, and since the projections of EF are now known those of M are found in the same manner as those of E and F.

Construction: (*c*)

Given: Lines AB and BC, Fig. 15.

Reqd: Bisector of angle " X."

The traces of T, the plane of AB and BC, are Tt' and

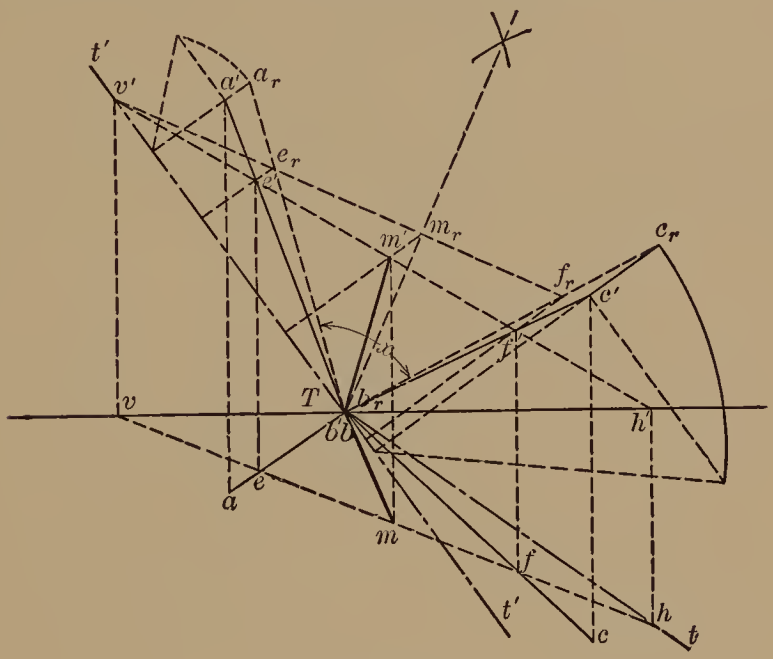


FIG. 14 *b*.

Tt. The plane is revolved about the V trace into V. A, B, and C fall at a_r, b_r, and c_r, respectively. In this revolution, it should be remembered that the triangle constructed with its right angle at b' gives, by its hypotenuse, the distance from B to the trace Tt'; also that the arc shows the angle Φ through which the plane is revolved. In revolving T about Tt' into V, and in counter-revolving it to its true position, every point in T travels through this same angle Φ.

The bisector of the revolved angle is $b_r n_r$. The projections of BM, the bisector, or in construction, the projections of the point M, since B is known, are found by reversing the construction whereby B was revolved. A perpendicular is drawn from m_r to the trace Tt' and an angle equal to the angle ϕ, through which B travels, is constructed with the intersection of this perpendicular and Tt' as a vertex. M

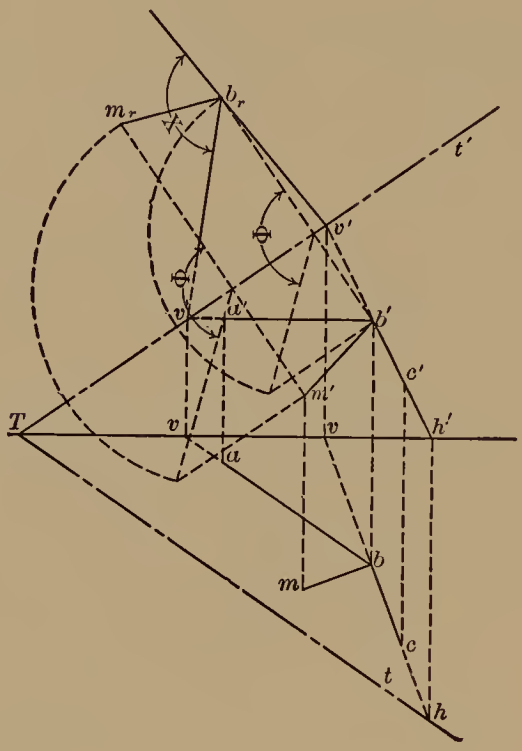


Fig. 15.

is then found in a position represented by the V projection m' on the perpendicular from n_r to Tt'. The H projection, m, is on the perpendicular to GL through m' and below GL a distance equal to the leg of the right triangle from m' to the arc.

(80) Problem 17.—**To pass a line through a given point making a given angle with a given line.**

Solution:

If a plane be passed through the given point and line and revolved about either trace into the corresponding coordinate plane the revolved position of the required line may be drawn through the revolved position of the point, making the given angle with the revolved position of the given line. If then the plane and the line just drawn be revolved back into their true positions, the required line is determined.

(81) Problem 18.—**To find the angle "X" which a given line makes with a given plane.**

Discussion:

The angle which a line makes with a plane is the angle which the line makes with its projection on that plane. If through some point of the line a perpendicular to the

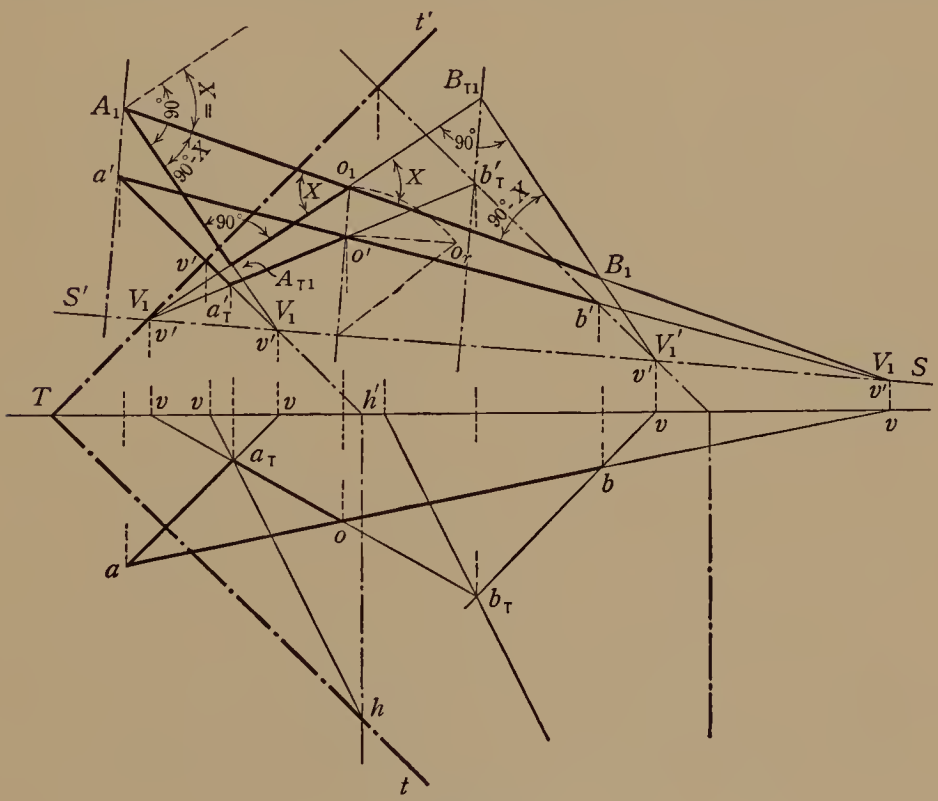


Fig. 16.

plane be passed, it will pierce the plane in the projection of the line; the given line, its projection on the plane, and the perpendicular form a right triangle of which one acute angle is the required angle "X," and the other the complement of angle "X."

Solution: (*a*)

Find the projection of the given line, AB, on the given plane, T, as explained in Problem 10; and measure the angle between the line and its projection, A_TB_T, on the plane as explained in Problem 16.

Solution: (*b*)

If nothing more than a measure of the angle is desired, a considerable portion of the construction shown in Fig. 16 may be eliminated by measuring the angle between the given line, AB and any one of the projecting perpendiculars, AA_T, to the given plane and subtracting this angle from 90°.

(82) Problem 18: Example 1.—**To pass a line through a given point making a given angle with a given plane.** [Converse of (81).]

Solution:

If through the given point a line perpendicular to the given plane be passed, the required line may then be passed through the given point, making with the perpendicular an angle equal to the complement of the given angle (Art. 80).

(83) Problem 19.—**To find the angles "θ" and "ϕ" which a given line makes with the H and V planes, respectively.**

Discussion:

Since "ϕ" and "θ" are the angles between the line and its H and V projections, either of the methods for finding the true length of a line will obtain these angles.

THE "Θ" CONE; THE "Φ" CONE; THE "Π" CONE

(84) A "Θ" **cone** is a right circular cone whose axis is perpendicular to H and whose base angle is equal to the angle "Θ" of a line or plane, either given or required.

A "Φ" **cone** is a right circular cone whose axis is per-

pendicular to V and whose base angle is equal to the angle " Φ " of a line or plane.

A " Π " **cone** is a right circular cone whose axis is perpendicular to P and whose base angle is equal to the angle " Π " of a line or plane.

(85) Problem 19: Example 1.—**To pass a line through a given point making given angles " Θ " and " Φ " with H and V.**

Solution:

If the given point be considered the apex of two right circular cones whose elements are of the same length, the one cone with its axis perpendicular to H and base angle equal to " Θ," the other with its axis perpendicular to V and base angle equal to " Φ," either of the lines of intersection of the surfaces of these two cones will be the required line.

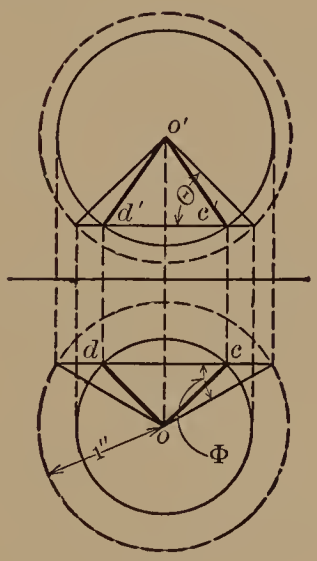


FIG. 17.

Construction:

Given: Point O, " Θ " =45°, " Φ " =30°, Fig. 17.

Reqd: Line through O at " Θ " and " Φ " to H and V.

The " Θ " cone is constructed with O as its apex and its elements 1 inch long. Its V projection is the isosceles triangle whose vertex is *o'* and base angle 45°; its H pro-

jection is the circle whose center is *o* and diameter equal to base of triangle. Then with O as the apex the " Φ " cone is constructed, its elements also 1 inch long; its H projection is the isosceles triangle whose vertex is *o* and base angle 30°, and its V projection the circle whose center is *o'* and diameter equal to base of 30° triangle. Since the elements of these two cones are equal, the circumferences of their bases lie on the surface of the same sphere (whose center is O and radius 1 inch) and intersect in points on the required lines of intersection of the cones. *cc'* and *dd'* are the projections of these points, and CO and DO are the required lines.

The substitution of Π for Φ or Θ with the corresponding substitution of P for V or H will furnish the solution of cases where the respective combinations of Π and Φ or Π and Θ are given.

(86) Problem 19: Example 2.—**To determine the second projection of a line when given one projection and the angle " Θ," " Φ " or " Π."**

Solution:

(For the angle " Θ.")

If any point of the given line be considered the apex of a right circular cone whose axis is perpendicular to H and whose base angle is equal to " Θ," the required line will be an element of this cone and is determined.

Construction:

Given: V projection of line AB and " Θ " =45°, Fig. 18.

Reqd: H projection of AB.

Any point O on AB is chosen as the apex of the " Θ " cone; the H projection of O may be at any distance above or below GL on the perpendicular from *o'* to GL. The V projection of the " Θ " cone is the isosceles triangle whose vertex is *o'*, base angles 45°, and the H projection is the circle whose center is *o* and whose diameter is equal to the base of the triangle. The second part of Fig. 17 illustrates the construction when Π is given instead of Θ.

Since the line is an element of this cone, the V projection of the point in which the element AB intersects the base of the cone is at c', where $a'b'$ intersects the base of the triangle; its H projection is at c, and the H projection of AB is determined.

If the angle "ϕ" or "Π" be given, the "ϕ" or "Π" cone must be used instead of the "θ" cone.

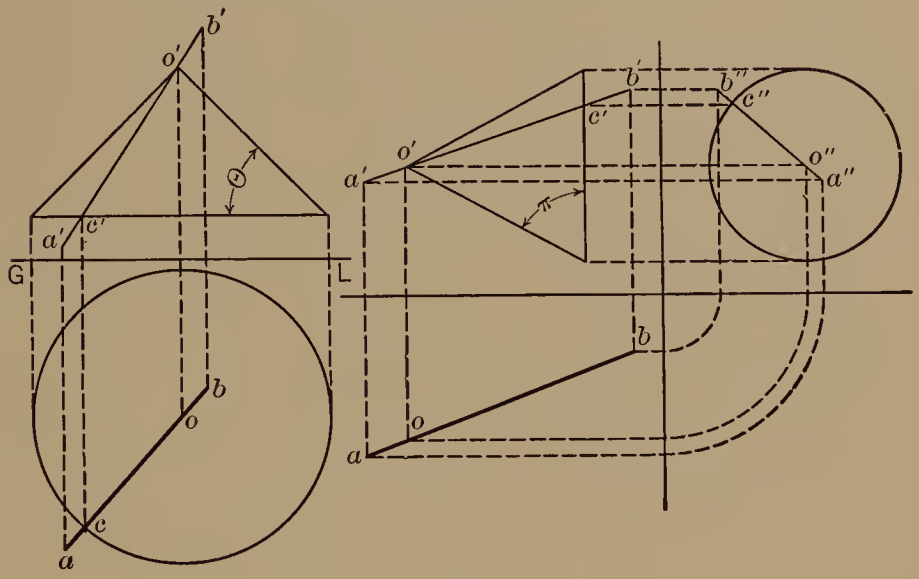


Fig. 18.

(87) Problem 20.—**To determine a line in a given plane making a given angle "θ," "Φ" or "Π," with H, V, or P.**

Solution:

(e.g., given angle "θ.")

If any point in the given plane be considered, the apex of a right circular cone whose axis is perpendicular to H, base angle equal to "θ," and base resting in H, either of the elements cut from this cone by the given plane will be the required line. Fig. 19.

(88) Problem 21.—**To find the angle "X" between two intersecting planes.**

Discussion: (*a*)

The angle between two intersecting planes is measured by the angle between two right lines, one in each plane and perpendicular to the line of intersection at the same point.

These two lines may be considered lines cut from the given planes by a third plane which is perpendicular to their line of intersection. The construction will be greatly simplified and the problem more easily understood if this auxiliary plane be drawn through some chosen point on the line of intersection rather than at random.

Solution: (*a*)

If through any chosen point on the line of intersection of the given planes a plane be passed perpendicular to this line, the angle between the lines cut by this plane from the given planes will be the required angle " X."

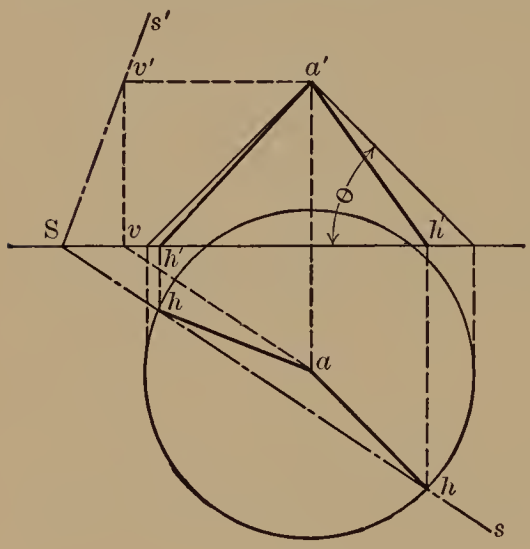


FIG. 19

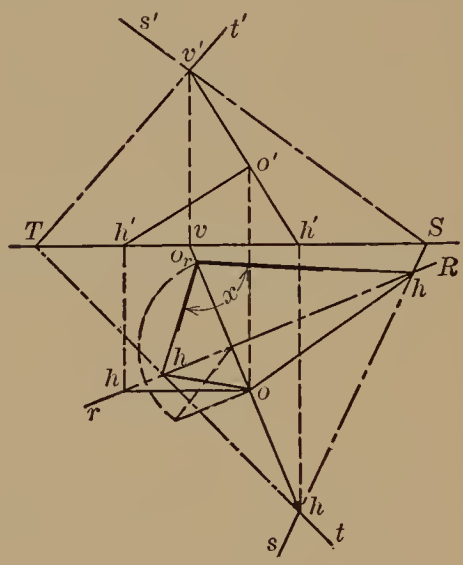


FIG. 20.

Construction: (*a*)

Given: Planes S and T, Fig. 20.

Reqd: Angle " X."

The projections of the line of intersection of planes S and T are *v'h'* and *vh*. Plane R is passed through any point O on this line perpendicular to VH; its H trace is *Rr*; the V trace is not required and is not drawn. Since the point O was chosen on the line of intersection of the two planes, and the plane R passes through O, the lines cut from S and T by R meet at O. O is therefore used in drawing these lines instead of their V piercing points. *oh* and *oh*

are then the H projections of the lines cut by R, from S and T. Plane R is revolved about *Rr* into H; O falls at o_r, and ho_rh is the true value of the required angle " X."

Discussion: (*b*)

If through any point in space lines be drawn perpendicular respectively to the two intersecting planes and a plane be passed through these perpendiculars, it will cut from the given planes right lines which with the two perpendiculars form a quadrilateral; two of the angles of this quadrilateral are right angles, therefore the remaining angles, i.e., the angle between the perpendiculars and the angle between the two lines lying in the planes, (angle " X "), are supplementary.

The two intersecting planes, however, make with each other two angles " X " which are supplementary; therefore the angle between the perpendiculars, being the supplement of one of these angles " X," is equal to the other.

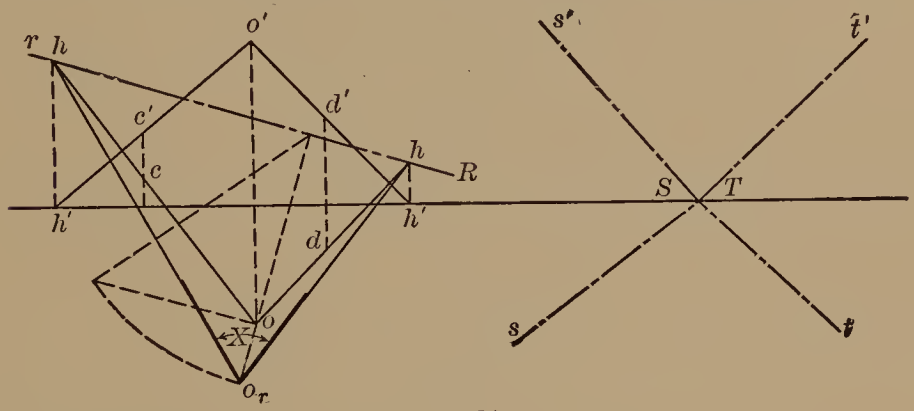


FIG. 21.

Solution: (*b*)

If through any chosen point in space two lines perpendicular respectively to the given planes be passed, the angle between these lines will be the required angle " X."

Construction: (*b*)

Given: Planes S and T, Fig. 21.

Reqd: Angle " X."

Point O is assumed in any convenient position and

through it OC is passed perpendicular to S, and OD perpendicular to T. *Rr* is the H trace of the plane of COD. Plane R is revolved about *Rr* into H. $ho_r h$ is the true value of COD, the required angle " X."

(89) Problem 21: Example 1.—**Through a given line in a given plane T, to pass a plane S making a given angle " X " with the given plane.**

Solution:

The given line in plane T is one line of the required plane S. If through any point O of the given line a plane be passed perpen-

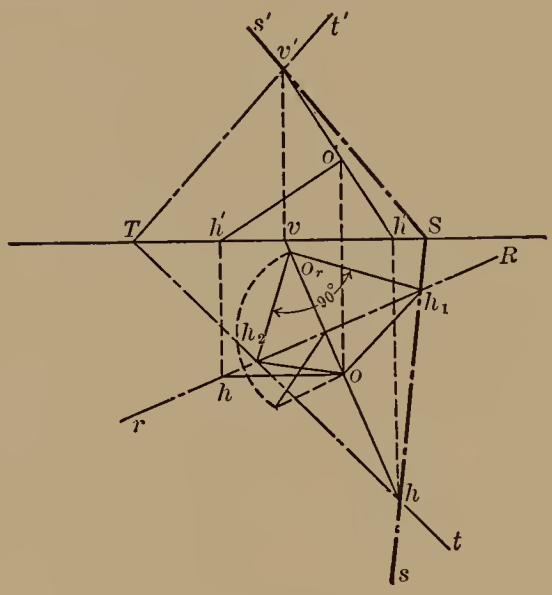


Fig. 22.

dicular to the line, a line in this plane passing through point O and making with the line of intersection of the two planes the given angle, is a second line of the required plane.

Construction:

Given: Plane T and line VH, Fig. 22.

Reqd: Plane through VH making 90° with T.

Through any point O on VH plane R is passed perpendicular to VH, Art. 57; its H trace *Rr* passes through *h* of the Vertical and is perpendicular to *vh*. oh_2 is the H pro-

jection of the line of intersection of R and T. After plane R is revolved about Rr into H, O falls at o_r and h_2o_r is the revolved position of the intersection of R and T. o_rh_1 is drawn through o_r making 90° with o_rh_2; since OH_1 is a line of plane R its H piercing point lies on Rr at the point h_1. The H trace of the required plane then passes through h_1 of line OH_1 and h of line VH; the V trace through v' of VH.

(90) Problem 22.—**To find the angle "Θ," "Φ," or "Π" of a given plane.**

Solution:

(To determine "Θ.")

If any point in the given plane be made the apex of a right circular cone which is tangent to the given plane, its axis per-

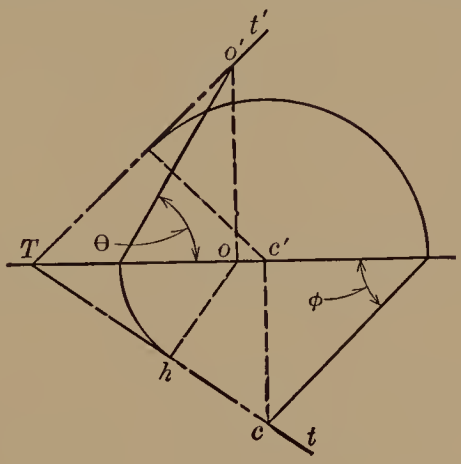


Fig. 23.

pendicular to H and its base resting on H, the base angle of this cone will be equal to the required angle "Θ."

For angle "Φ," substitute in the above analysis, H for V, V for H, and "Φ" for "Θ."

For angle "Π" substitute P for H and "Π" for "Θ."

Construction:

Given: Plane T, Fig. 23.

Reqd: "Θ" and "Φ."

For convenience, point O on the V trace is chosen as the apex of the right circular cone; the center of its base is at o

and the radius of the base is the perpendicular distance from *o* to *Tt*. The H projection of this cone is then the circle whose center is *o* and radius *oh*; the V projection is the isosceles triangle whose vertex is *o'* and base equal to the diameter of circle. The base angle of this triangle is the required angle " Θ."

(91) Problem 22: Example 1.—**To pass through a given point a plane which will make given angles " Θ " and " Φ " with H and V.**

Solution:

If any point on GL be considered the center of a sphere of any radius, and two right circular cones be constructed tangent to this sphere, the first with its axis in V and perpendicular to H, the base resting in H, and the base equal to " Θ," the second with its axis in H and perpendicular to V, its base in V and base angle equal to " Φ," the two planes which may be passed tangent to both of these cones make with H and V the given angles Θ and Φ. A plane through the given point parallel to either of these planes is the required plane.

Construction:

Given: Angles " Θ " and " Φ," and point *M*, Fig. 24.

Rexd: Plane S.

The center of the sphere is assumed at O and the diameter any desired length. The projections of the first cone (axis in V and perpendicular to H, base in H, and base angle equal to " Θ ") are: V projection, the triangle *v'd'c'* tangent to the projection of the sphere; H projection, the circle whose center is *v* and radius *vd*. Since the apex of this cone is in V and its base in H, the V trace of any plane tangent to it will pass through the V projection of the apex and its H trace will be tangent to the circle which is the H projection of the base of the cone.

The projections of the second cone (axis in H and perpendicular to V, base in V and base angle equal to " Φ ") are: H projection, the triangle *hab* tangent to the H projection of the sphere; V projection, the circle whose center is *h'* and radius *h'b'*. Since the apex of this cone is in H and

the base in V, the H trace of any plane tangent to it will pass through the H projection of the apex and its V trace will be tangent to the circle which is the V projection of the base of the cone.

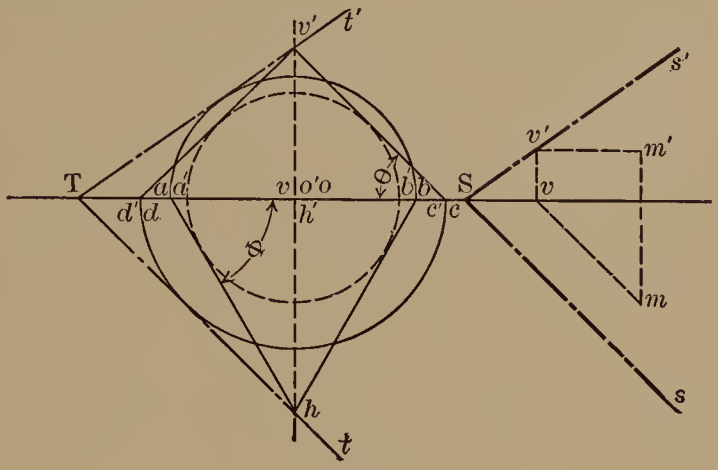


FIG. 24.

From the two deductions above, the V trace of T, one of two planes tangent to both cones, passes through v' tangent to the small circle whose diameter is $a'b'$; the H trace passes through h tangent to the large circle whose diameter is cd. The required plane S passes through the given point M and is parallel to T.

(92) Problem 22: Example 2.—**To determine the second trace of a plane, when given one trace and the angle "θ" or "Φ."**

Solution:

(When given the V trace and "θ.")

If any point on the given V trace be considered the apex of a right circular cone which is tangent to the plane, its axis perpendicular to H, base angle equal to "θ," and base resting in H, the V trace of the plane passes through the apex of this cone and the H trace is tangent to the circumference of the base.

(When given the V trace and "Φ.")

If any point on the given V trace be considered the point of tangency of this trace with the base of the "Φ" cone which is tangent to the plane, axis in H and perpendicular to V, base resting

in V, the required H trace will pass through the apex of this cone and the intersection of the given trace with GL.

Construction:

Given: V trace of plane T and angle " θ," Fig. 25.

Reqd: H trace of Plane T.

Any point *o'* on *Tt'* is selected as the V projection of the apex of the " θ " cone; *o'c'd'* is the V projection of this

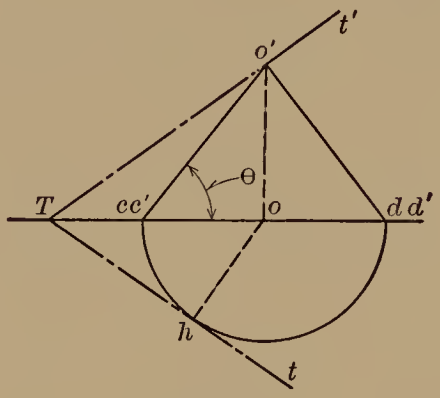


Fig. 25.

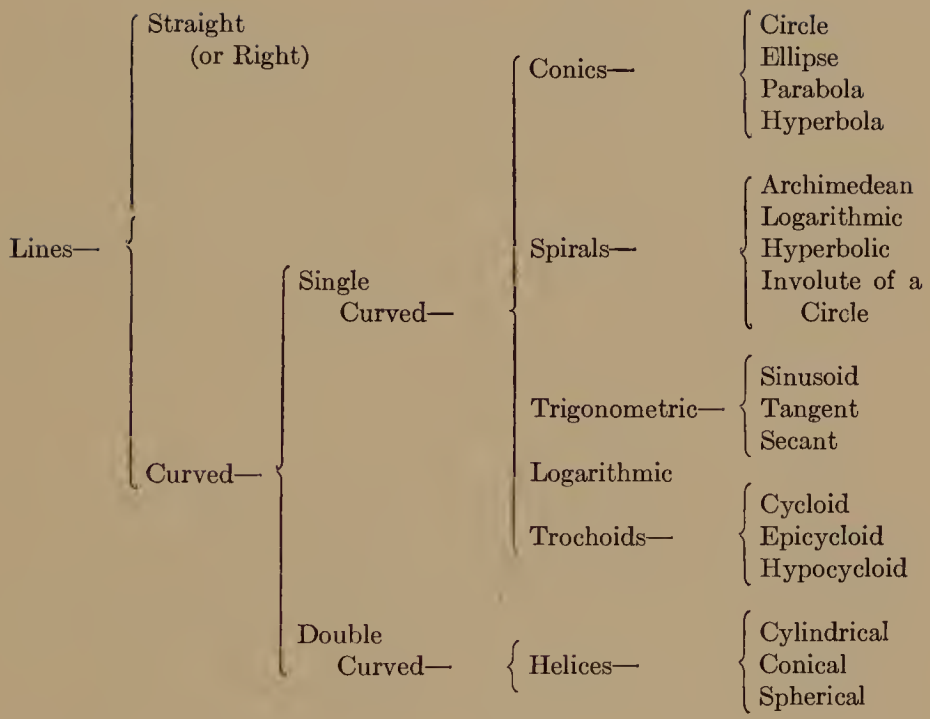
cone, the circle *chd* the H projection. Since the base of the cone rests in the H plane, the required H trace *Tt* is tangent to the circumference of the base. *Tt* is therefore drawn tangent to the circle *chd.*

PART II

CHAPTER VI

LINES AND SURFACES

CLASSIFICATION OF LINES



LINES

(93) A straight line has been defined in Art. 22. A curved line is generated by a point moving in a constantly changing direction. A line of *single curvature*, commonly called a **plane curve**, is a line all points of which lie in a plane. A line of *double curvature*, commonly called a **space**

curve, is a line only three successive points of which lie in the same plane.

(94) *The projections of a curve are composed of the projections of the component points of the curve.* If all the points of a curve be projected upon the H plane and a smooth curve be drawn through these projections, the resultant curve is the H projection of the given curve. Compare Art. 22. If through the V projections of all the points of a curve a smooth curve be drawn, this curve is the V projection of the given curve.

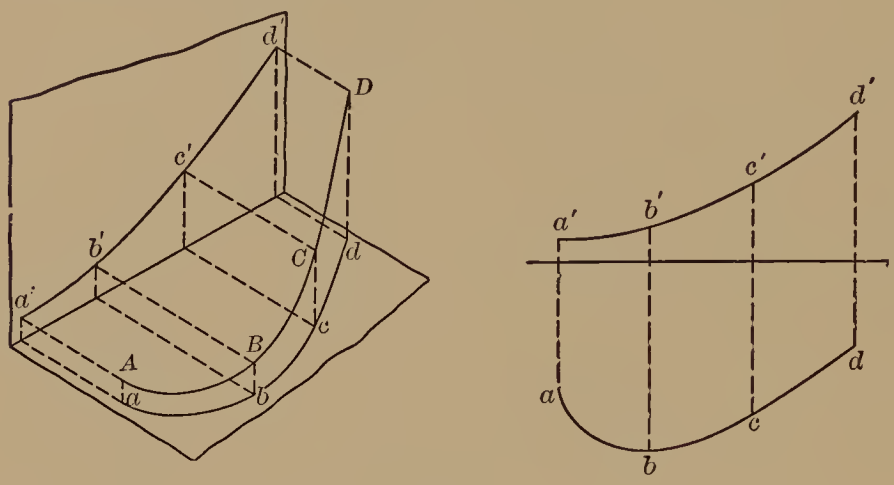


FIG. 1.

(95) If the two projections of a curve are given, the curve is generally completely determined. For if from some point in one projection of a curve a perpendicular to GL be drawn until it intersects the other projection of the curve, Fig. 1, these two points are projections of a definite point in space.

(96) If the plane of a curve of single curvature is perpendicular to H, the projection of the curve on that plane is a right line coinciding with the H trace of the plane. In like manner the V projection of a curve may be a right line.

If the plane of the curve is perpendicular to GL both the H and V projections of the curve are right lines and perpendicular to GL.

If the plane of a curve is parallel to either H or V its projection on that plane is an identical curve, the other projection a right line parallel to GL.

TANGENTS AND NORMALS TO CURVES OF SINGLE CURVATURE

(97) A line which lies in the plane of a given curve and has two consecutive points in common with it, is called a **tangent** to that curve. A line lying in the plane of the curve and perpendicular to the tangent at one of these consecutive points of tangency is a **normal** of the curve.

If two curves lie in the same plane and have two consecutive points in common they are called **tangent curves.**

(98) *If two lines are tangent in space their projections are tangent.* For if H projecting lines be passed through the several points of the two curves, only the lines from the two common consecutive points coincide; the H projections of the curves can therefore have only two points in common and these points consecutive; the H projections of the given curves are hence tangent. The V projections are likewise tangent, and the *projections of the consecutive points of tangency are the points of tangency of the projections.*

SURFACES

(99) Surfaces in general are conceived as being generated by a line moving according to some law. They are called **ruled surfaces** if they can be generated by a *right line,* and **double curved surfaces** if the moving line cannot be other than a *curve.*

The moving line which generates the surface is called the **generatrix.** Any position of the generatrix is called an **element** of the surface.

RULED SURFACES

(100) **Ruled surfaces** are generated by a right line moving according to some law.

a. **Planes** are generated by a right line moving so as to

touch another right line and having all of its positions parallel.

b. **Single curved surfaces** are generated by a right line, moving so that it constantly touches a given curve.

Single curved surfaces are divided into three classes, two of which are:

Cylinders—All positions of the generating line *parallel.*

Cones—All positions of the generating line *intersecting in a common point.*

CLASSIFICATION OF SURFACES

- SURFACES
  - Ruled—
    - Plane
    - Single Curved—
      - Cylinders
      - Cones
      - Convolutes
    - Warped—
      - Hyperbolic Paraboloid
      - Hyperboloid of Revolution of One Nappe
      - Helicoid
      - Conoid
      - General Forms
        - 3 Linear Directrices
        - 2 Linear Dir. and a Plane Dir.
  - Double Curved—
    - Sphere
    - Ellipsoid—
      - Spheroid (Ellipsoid of Revolution) Two Axes Equal
        - Prolate
        - Oblate
      - General Form Three Axes Unequal
    - Paraboloid—
      - Paraboloid of Revolution
      - Elliptical Paraboloid
    - Hyperboloid of Two Nappes—
      - Hyperboloid of Revolution of Two Nappes
      - Elliptical Hyperboloid
    - Torus

CYLINDERS

(101) A cylinder may be generated by a right line moving so that it is always parallel to its first position and constantly touches a given curve. If the cylinder be inter-

sected by a plane, the curve of intersection may be considered the **base** of the cylinder.

If we wish to consider a definite portion of the surface, a second plane may be passed parallel to the first, intersecting the cylinder in a second base, one being called the **lower base,** the other the **upper base.** The portion of the surface between these two bases is termed **" the cylinder."**

A line through the center of the base and parallel to the elements is termed **the axis** of the cylinder.

A cylinder whose elements are *perpendicular* to the plane of the *base* is termed a **right cylinder.**

A cylinder whose elements are *oblique* to the *plane* of the *base* is termed an **oblique cylinder.**

A plane *perpendicular* to the *elements* of a cylinder intersects it in a **right section.**

A cylinder is represented in general by the projection of a base, its axis and the limiting elements in each projection.

The shape of the base and the relative position of the axis of a cylinder are usually indicated by designating it as a right-circular cylinder, oblique-elliptical, etc.

VISIBILITY

(102) The various surfaces treated in Descriptive Geometry are considered opaque. Only half of such surfaces as the sphere, spheroid, etc., are visible in each projection; and points or lines on these surfaces may be wholly or in part on the invisible portion in one or both projections. When two surfaces are involved in the same problem, points or lines that are on both, for instance their line of intersection, may be on the near side of one surface but concealed wholly or in part by the other. The unseen part of any such line should be so designated by the proper convention.

In determining which portions of a line of intersection should be represented as visible and which invisible in the V projection, reference to the H projection must often be made; that is, it may be easier to determine from the H

projection which part of a line of intersection is on the front and which on the back of a given surface. For example, in the case of the ellipsoid, Fig. 2, since it has its axis perpendicular to H, all of the surface before the median plane X is visible in the V projection, and all above the median plane Y is visible in the H projection. Further, the projections of the points where a line on the surface passes from the visible to the invisible must be on the contour of that projection. (See e' and f', Fig. 2.) The V projection of the line must therefore touch the contour at that point, being either tangent to it, or forming a cusp at that point.

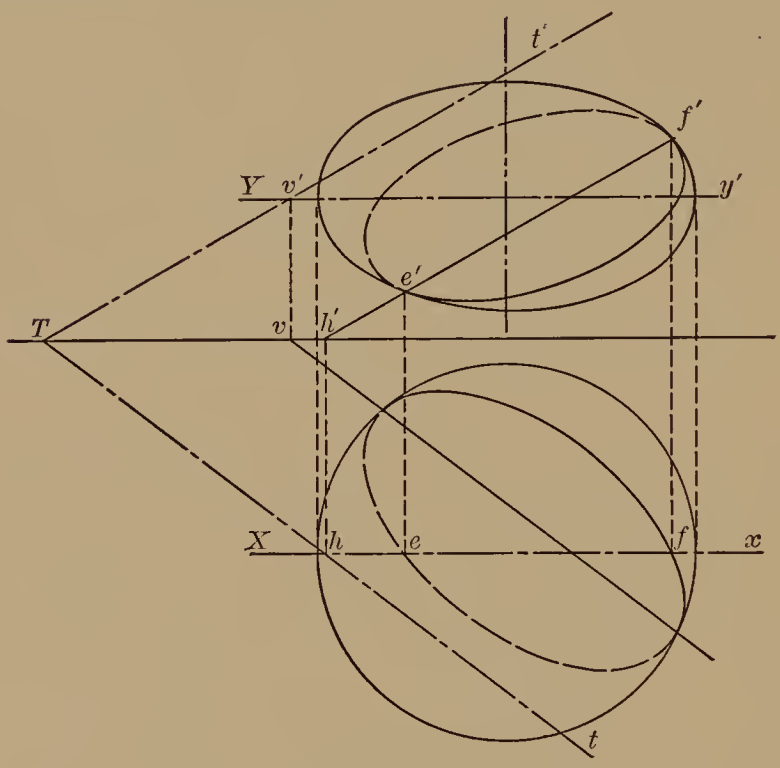


Fig. 2.

In dealing with ruled surfaces the determination of visibility of points may often be accomplished by determining the visibility of the elements on which the points lie. If a point on the intersection of two surfaces is on elements

of both surfaces that are visible from the front, this point is indicated as visible in the V projection. If an invisible element of either surface passes through the point, the point is invisible in that projection.

(103) Problem 23.—**To assume a point on the surface of a given cylinder.**

Solution:

An element of the surface may be assumed by passing through any point on the given base a line parallel to the other elements, or axis, of the surface. Any point on this element is a point on the surface of the cylinder.

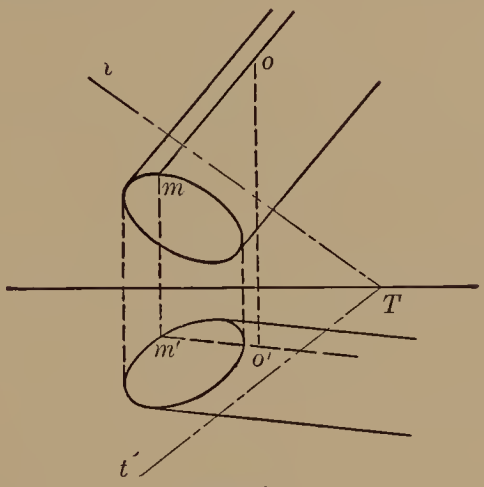


Fig. 3.

If one projection of the point is given, the corresponding projection of the element through this point may be drawn parallel to the axis. Since this element must intersect the curve of the base at a point in the plane of the base, the element and the point on the element are completely determined, Fig. 3.

CONES

(104) A cone is generated by a right line moving so that it constantly touches a given curve and passes through a fixed point not in the plane of the curve.

The given curve is called the **directrix,** the given point the **apex,** and the different positions of the generating line the **elements** of the cone.

If a given plane intersects all of the elements of the cone,

the curve of intersection is called the base of the cone in that plane.

The right line joining the apex with the center of the base is called the axis of the cone.

A cone is called a right cone or an oblique cone, depending on whether the axis is perpendicular to or oblique to the plane of the base.

The shape of the base and relative position of the axis of a cone are usually indicated by designating it as right-circular, oblique-elliptical, etc.

(105) Problem 24.—**To assume a point on the surface of a given cone.**

Solution:

With slight rewording to indicate that all elements of a cone intersect at the apex, the solution for assuming a point on a cylinder applies equally well to the cone.

DOUBLE CURVED SURFACES

(106) **Double curved surfaces** can be generated only by curves *of single curvature* moving according to some law.

A double curved **surface** of **revolution** may be generated by revolving a given curve about a given right line as an axis. The plane of the given curve may or may not contain the axis of revolution. If it contains the axis, then a plane through the axis intersects the surface in a curve identical with the generating curve.

If a *circle* be revolved about one of its *diameters* as an axis, the surface generated will be a **sphere.**

If an *ellipse* be revolved about its *major axis* the surface generated will be a **prolate spheroid.** If revolved about the *minor axis* the surface will be an **oblate spheroid.**

(107) Problem 25.—**To assume a point on a surface of revolution.**

Solution:

If an auxiliary plane be passed perpendicular to the axis of revolution within the proper limits it will intersect the surface in a circle. Any point of this circle will lie on the given surface.

CHAPTER VII

TANGENT PLANES

DEFINITIONS

(108) A line is tangent to a surface at a given point if it is tangent to any line on the surface at that point.

A plane tangent to a surface at a given point is the locus of all right lines tangent to the surface at that point.

If a plane is tangent to a surface at a given point, then any other plane passed through this point cuts a straight line from the tangent plane and a line from the surface such that these two lines are tangent at the given point.

PLANES TANGENT TO SINGLE CURVED SURFACES

(109) Problem 26—**To pass a plane through a given point on a given cone or cylinder tangent to the surface.**

Discussion:

In the case of all single curved surfaces, a tangent plane has one line in common with the surface, i.e., an element. This element, and a line tangent to the surface at any point on the element, determine the plane.

Solution:

The element of the surface through the given point provides one line for the determination of the plane. The line tangent to the circumference of the base through the point of intersection of the element with the base is a second line. The required plane is determined by this tangent and the element.

Construction:

Given: Cone X and the V projection of the point O on cone, Fig. 1. (Base in V projecting plane.)

Reqd: Plane through O tangent to cone.

The V projection of an element of the cone is drawn through o' and x' and produced until it intersects the V

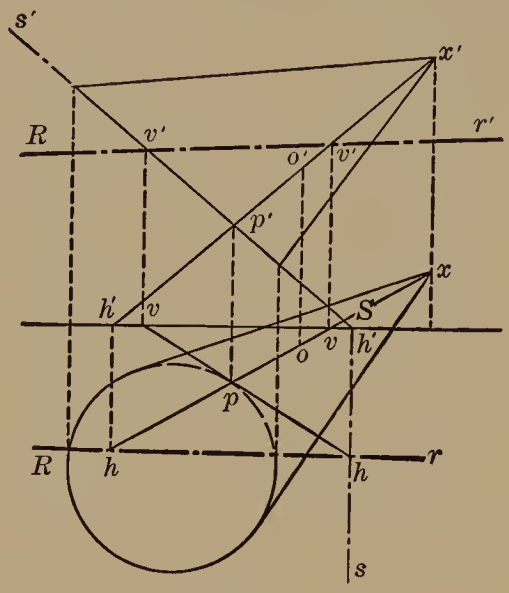


Fig. 1.

projection of the base at p'; p' is the V projection of the point in which the element OX pierces the plane of the base. its H projection is p. The H projection ph of the line tangent to the H projection of the base is drawn through p; since this tangent line lies in the plane of the base its V projection $p'h'$ coincides with the V projection of the base of the cone. The required plane R passes through the two intersecting lines OP and PH.

(110) Problem 27.—**To pass a plane tangent to the surface of a given cone through a given point in space.**

Solution:

A right line through the given point and the apex of the cone is one line of the required plane. If this line be extended until it pierces the plane of the base of the cone and through this piercing point a line be passed tangent to the circumference of the base, this

tangent line and the line through the given point determine the required plane. The element through the point of tangency of the second line to the curve of the base is the element of tangency and must also lie in the tangent plane.

Construction:

Given: Cone X and point O, Fig. 2.

Reqd: Plane through O tangent to cone.

A right line is passed through O and X. This line OX pierces the plane of the base at P, Art. 63. The line PC is passed through P tangent to the base of the cone. The required plane T passes through PX and PC.

Note.—If the given surface be a cylinder the above analysis applies equally to it since a cylinder may be considered a cone with its apex infinitely distant, Art. 96. The

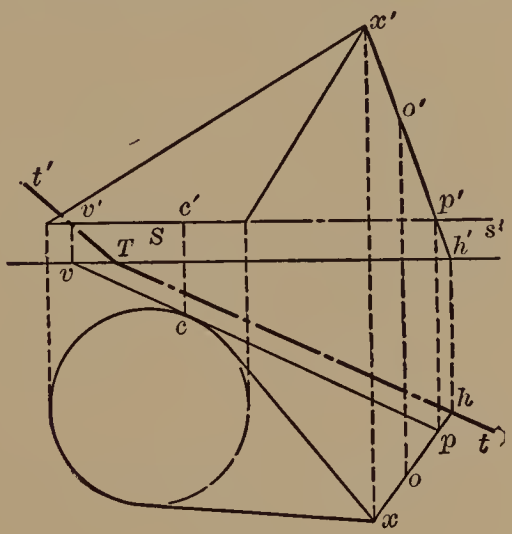


Fig. 2.

line through the given point is then parallel to the elements of the cylinder.

(111) Problem 28.—**To pass a plane tangent to a given cone and parallel to a given line.**

Solution:

A right line through the apex of the cone parallel to the given line is one line of the required plane. If this line be extended until it pierces the plane of the base of the cone, a right line through this piercing point tangent to the base of the cone is a second line of

the required plane. The required plane is then determined by this tangent line and the line passed through the apex. The element through the point of tangency of the second line to the curve of the base is the element of tangency and must also lie in the tangent plane.

Construction:

Given: Cone X and line AB, Fig. 3.

Reqd: Plane tangent to cone and parallel to AB.

The line XP is passed through the apex X parallel to the given line AB. P is the piercing point of this line in the

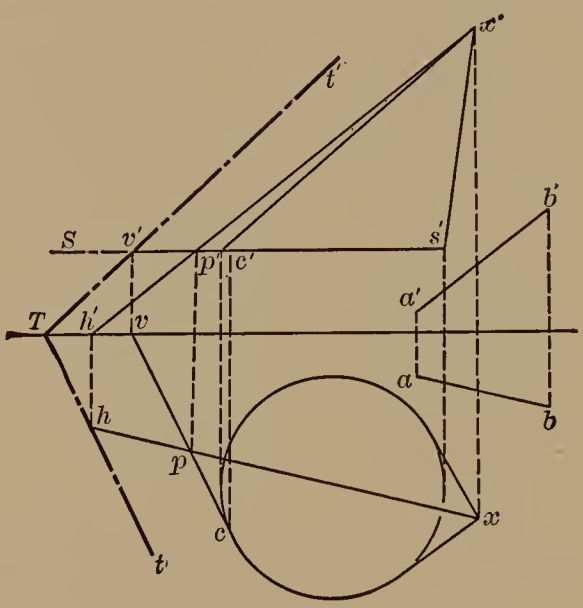


Fig. 3.

plane S of the base of the cone, Art. 55. The line PC is passed through P tangent to the base of the cone at C. The required plane T passes through the two lines PX and PC.

(112) Problem 28: Example 1.—**To pass a plane tangent to a given cylinder and parallel to a given line.**

Solution:

If through any point of the given line a line parallel to the elements of the cylinder be passed, the plane of these two lines is parallel to the required plane. A line tangent to the base of the cylinder and parallel to the line of intersection of the plane just determined

with the plane of the base is one line of the required plane. The element through the point of tangency is another. The required plane is determined by these two intersecting lines.

PLANES TANGENT TO DOUBLE CURVED SURFACES

(113) Problem 29.—**Through a given point on a given surface of revolution to pass a plane tangent to the surface.**

Discussion:

If through the given point two right lines be passed tangent to the surface, they determine the required tangent plane. By Art. 108, any two tangents may be used. Two of these tangent lines, however, are more easily determined than others and are generally used.

Solution:

A plane through the given point perpendicular to the axis of revolution intersects the surface in a circle. A line through the given point tangent to this circle is one of the two required tangent lines. If now a right circular cone be constructed with its apex on the axis of revolution and tangent to the surface along this circle, the element of this cone passing through the given point is tangent to the surface of revolution and is a second line of the required plane. The plane passing through these two tangent lines is the required plane.

Construction:

Given: Spheroid and point P on spheroid, Fig. 4.

Reqd: Plane through P tangent to spheroid.

Plane S is passed through point P perpendicular to the axis of revolution. It intersects the surface in a circle whose H projection is the chord oo and whose V projection is the circle with its center at x' and diameter equal to oo. The line PH is passed through P tangent to this circle. A right circular cone is constructed with its apex X on the axis of revolution, and tangent to the spheroid along the circle cut from the surface by plane S. Its H projection is the isosceles triangle xoo, the sides xo and xo being tangent to the ellipse at o and o; the V projection is the circle $o'p'o'$. Since all of the elements of this cone are tangent to the

spheroid, the element XP is the second of the two required tangent lines. The required plane T passes through the tangent lines PH and PX.

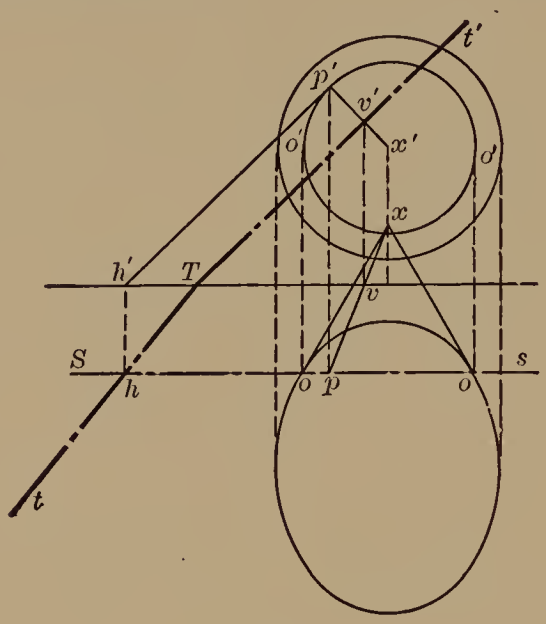


Fig. 4.

(114) Problem 29, Example 1.—**Through a given point on a given sphere to pass a plane tangent to the sphere.**

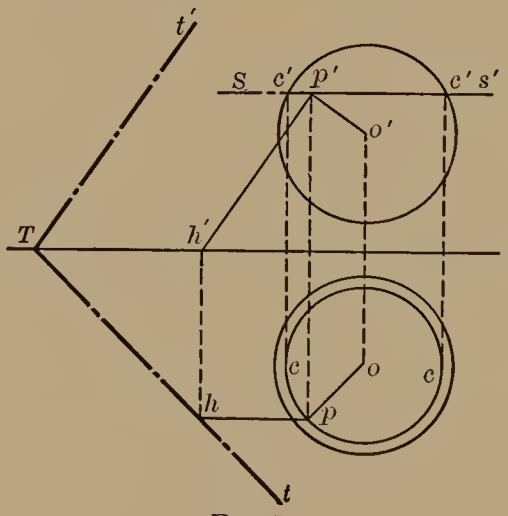


Fig. 5.

Solution:

If a plane be passed tangent to a sphere the radius of the sphere which passes through the point of tangency is perpendicular to

the tangent plane. Conversely, the plane through any given point on a sphere perpendicular to the radius through that point is the required tangent plane.

Construction:

Given: Sphere O and point P on sphere, Fig. 5.

Reqd: Plane through P tangent to sphere.

The V projection p' of point P is given. The H projection is determined by passing through P a plane S parallel to H. S intersects the sphere in a circle whose V projection is $c'c'$ and H projection a circle with center at o and diameter equal to $c'c'$. Since point P lies on this circle its H projection is at p. OP is the desired radius of the sphere. The required plane T passes through point P perpendicular to this radius OP, Art. 57.

CHAPTER VIII

INTERSECTION OF SURFACES

(115) Intersecting surfaces but rarely intersect in right lines. Various methods may be employed to locate a number of points common to the intersecting surfaces, that is, on their line or lines of intersection. Some of these methods are decidedly easier of execution than others. Even though it has been stated that this book is an outline of exercises in the application of Plane and Solid Geometry to the representing of objects common to the engineer, it is felt that the simpler methods available for the determining of the intersections of surfaces are sufficiently complex to furnish the desired mental exercise, and these are used in preference to the more complex.

A simple and generally used method of determining the intersection of surfaces involves the passing of planes in such a manner as to intersect the surface in lines which project on the coordinate planes as either right lines or circles. In applying this method the following facts of Solid Geometry should be remembered:

a. Secant planes passing through the apex of a cone intersect the surface in elements.

b. Secant planes parallel to the axis of a cylinder intersect the surface in elements.

c. Secant planes parallel to the base of a cone or cylinder intersect the surface in curves similar to or identical with the base.

d. Secant planes perpendicular to the axis of a surface of revolution intersect that surface in one or more circles.

e. Secant planes through the axis of a surface of revolu-

tion intersect the surface in curves identical with the generating curve.

In certain types of problems none of the above methods will apply, and the solution then involves sections which are not so simple.

NUMBER OF CURVES OF INTERSECTION

(116) Two surfaces may intersect in such a way as to give either one or two closed curves, or either one or two open curves, these curves being of various shapes according to the type of surfaces considered. Various types of curves may be noted in the following problems. Figs. 1 and 2 illustrate convenient methods for numbering elements in obtaining lines of intersection.

PLANES AND SINGLE CURVED SURFACES

(117) Problem 30.—**To find the line of intersection of a given plane with a given cone or cylinder.**

Discussion:

The application of the general rule, Art. 115, to this case is most easily made by using planes which intersect the cone in elements. The construction is then identical with that of Problem 8, Art. 63.

Solution:

Elements of the cone or cylinder pierce the given plane in points common to both, hence on their line of intersection. The line connecting these points is the required line of intersection.

Construction:

Given: Plane T and cone X, Fig. 3.

Reqd: Line of intersection.

1. A number of elements of the cone are assumed at regular intervals and in sufficient number to insure an accurate curve. (It is a good practice to number these elements carefully.) The piercing point of each element with the given plane is found, as indicated for the two in

Fig. 3. The line of intersection is the line connecting these points.

Note.—In each of the figures in this chapter, only one auxiliary plane is passed, and the points found by the elements it contains are indicated. The complete construction involves simply a repetition of the same operation with other planes.

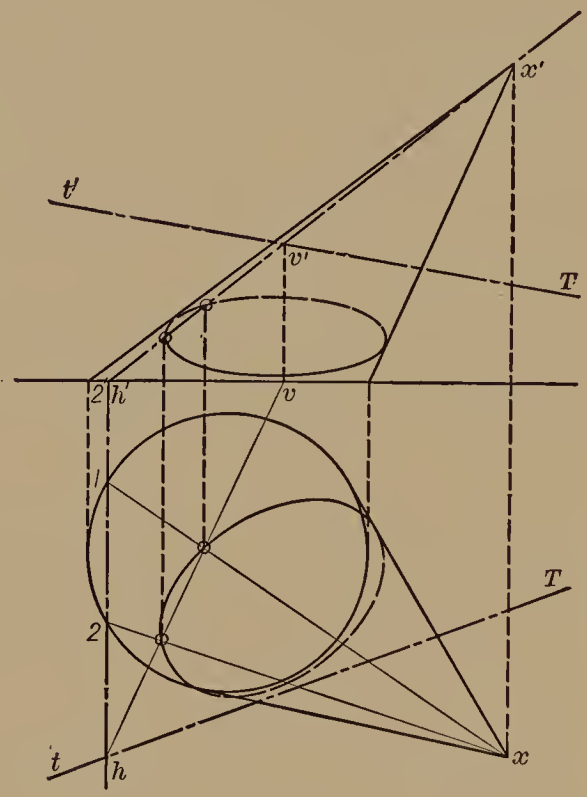


Fig. 3.

PLANES AND DOUBLE CURVED SURFACES

(118) Problem 31.—**To find the line of intersection of a given plane and a given surface of revolution.**

Discussion:

In this and the succeeding problems the construction follows the general method of first passing auxiliary planes and then finding the lines that they cut from each surface.

Solution:

Planes perpendicular to the axis of the surface intersect the surface in circles and the given plane in right lines. The points of intersection of these right lines and circles are common to the surface and the given plane, hence are points of the required line of intersection.

Construction:

Given: Plane T and spheroid, Fig. 4.

Reqd: Line of intersection.

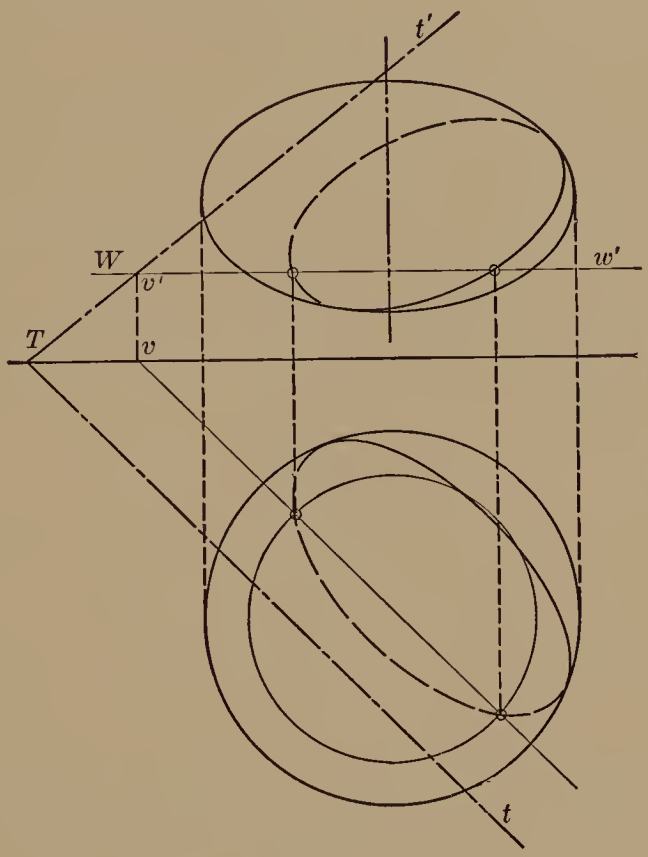


Fig. 4.

A series of planes are passed perpendicular to the axis of the spheroid, sufficiently close together to insure an accurate curve; since the axis is perpendicular to H all of these planes are parallel to H and their lines of intersection with plane T are horizontals of plane T. Each plane intersects

the ellipsoid in a circle whose V projection coincides with the trace of the plane and whose H projection is a circle with diameter equal to the chord of the ellipse. The points of intersection of these circles and the corresponding Horizontals of plane T are points on the required line of intersection. One plane is shown in Fig. 4, determining two points. Other points may be found similarly, and the line of intersection traced through them.

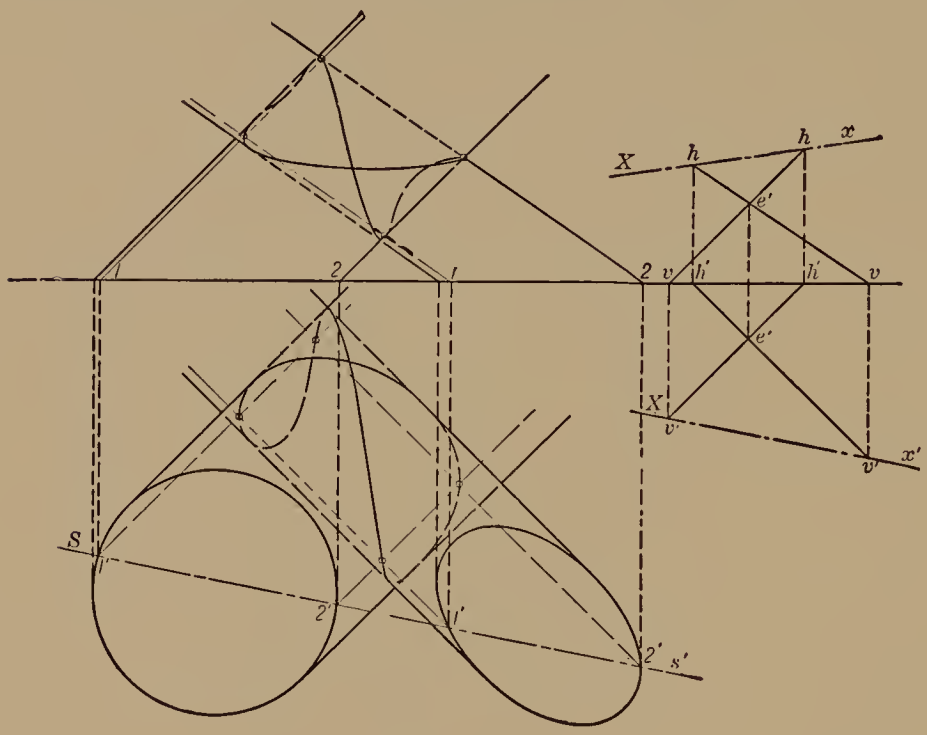


Fig. 5.

INTERSECTIONS OF SINGLE CURVED SURFACES

(119) Under this classification we may include the combination of cone with cone, cone with cylinder, or cylinder with cylinder. In any case, the method usually employed is to pass the auxiliary planes in such a manner as to intersect the surfaces in elements. Application of the principles given at the beginning of this chapter require, then, that each auxiliary plane contain the apex of the cone and be parallel to the elements of the cylinder, if both are

involved; with two cones, each plane must contain the apexes of both; in the case of two cylinders, the planes must be parallel to the elements of each cylinder.

(120) Problem 32.—**To find the line of intersection of two given cylinders.**

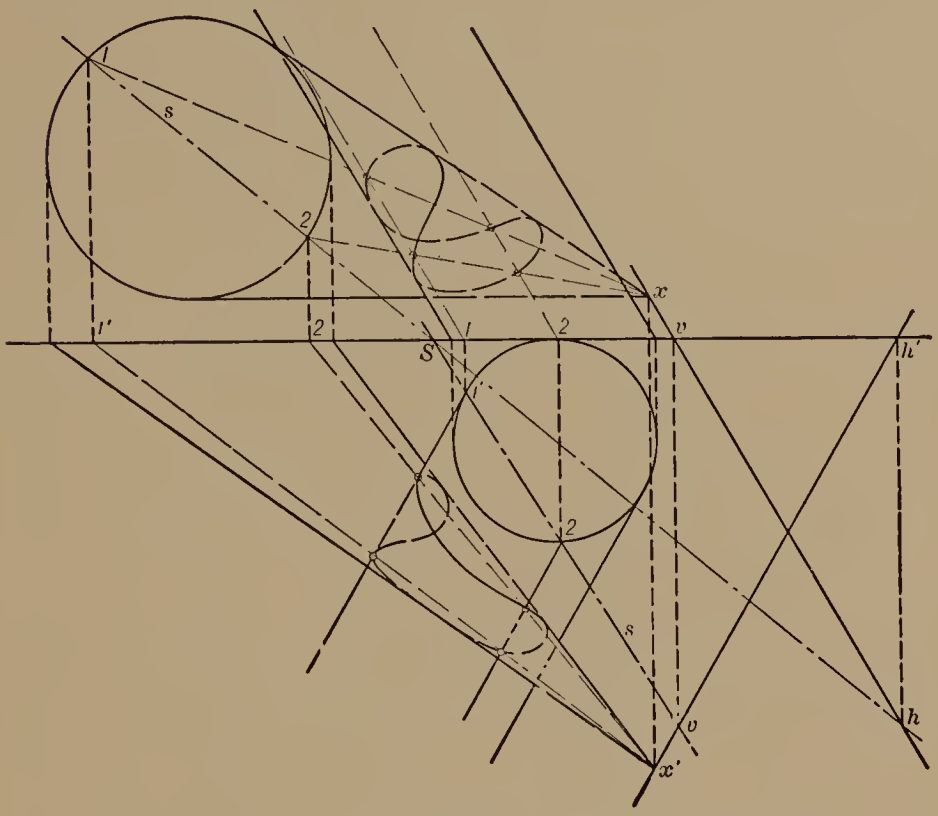


Fig. 6.

Solution:

Common secant planes parallel to the elements of the two given cylinders intersect the given cylinders in elements which intersect in points on the required line of intersection.

Construction:

Given: Two cylinders, Fig. 5.

Reqd: The line of intersection

Through any auxiliary point E, two lines are drawn, parallel to the elements of the two cylinders respectively.

Plane X is then passed through these two intersecting lines. Each auxiliary plane is then passed parallel to this plane (Art. 41). One such plane is S, which intersects the cylinders in two elements each. Each element of the one cylinder intersects both elements of the other cylinder giving

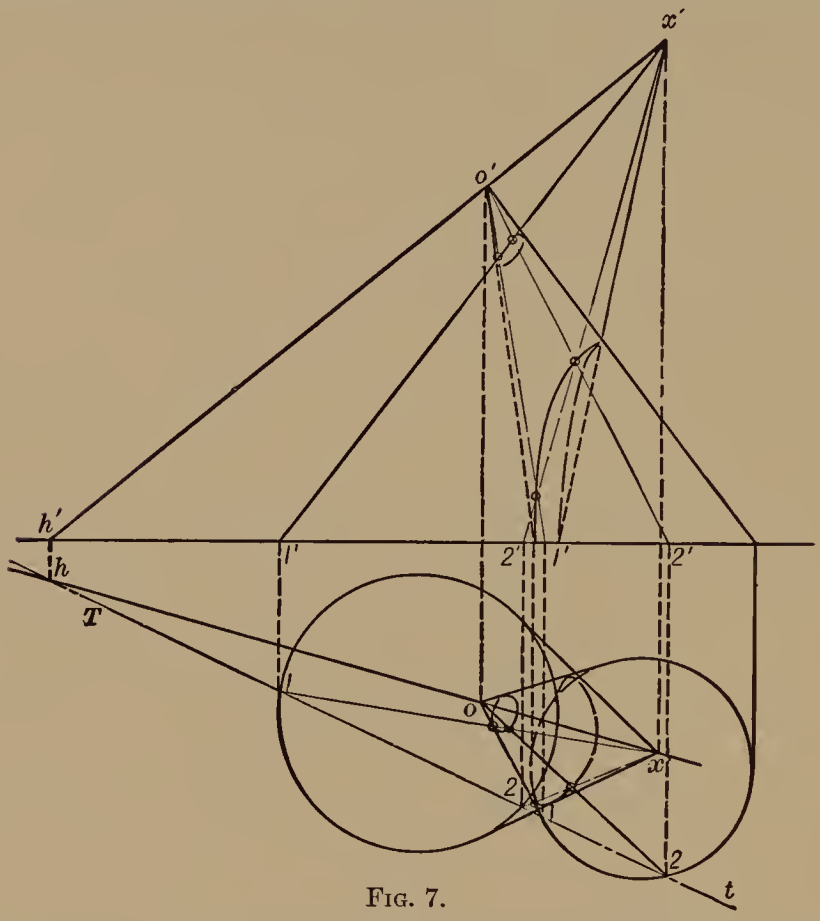


Fig. 7.

four points of the required line of intersection. Other points are determined by additional auxiliary planes and the required line of intersection traced through them.

(121) Problem 33.—**To find the line of intersection of a given cone with a given cylinder.**

Solution:

Common secant planes through the apex of the cone parallel to the elements of the cylinder intersect both surfaces in elements which intersect in points on the required line of intersection.

Construction:

Given: Cone and cylinder, Figs. 2, 6, 8, 9, 10.

Reqd: Line of intersection.

The line XH is passed through the apex of the cone, parallel to the elements of the cylinder. Each auxiliary plane must contain this line. One such plane is S which intersects the cone and cylinder in two elements each. Each element of the cone intersects both elements of the cylinder, giving four points of the required line of intersection. Other points are determined by additional auxiliary planes and the line traced through them.

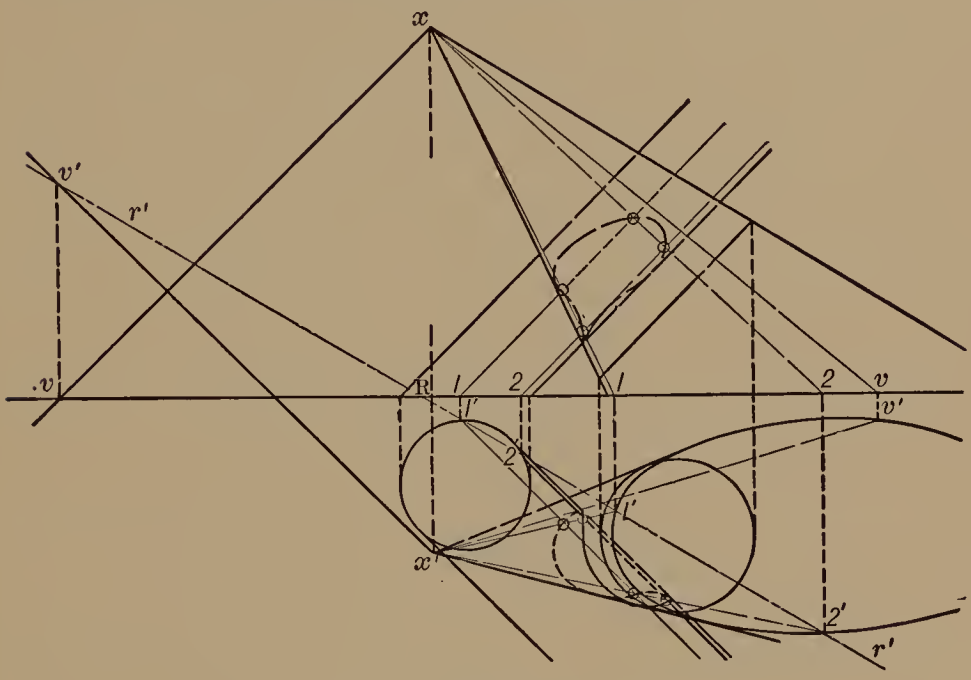


Fig. 8.

(122) Problem 34.—**To find the line of intersection of two given cones.**

Solution:

Common secant planes through the apexes of the given cones intersect the cones in elements which intersect in points on the required line of intersection.

Construction:

Given: Cones X and O, Fig. 7.

Reqd: Line of intersection.

The line OX is passed through the apexes of the cones; the H projection of its H piercing point is at *h*. The H traces of all planes passed through line OX pass through *h*.

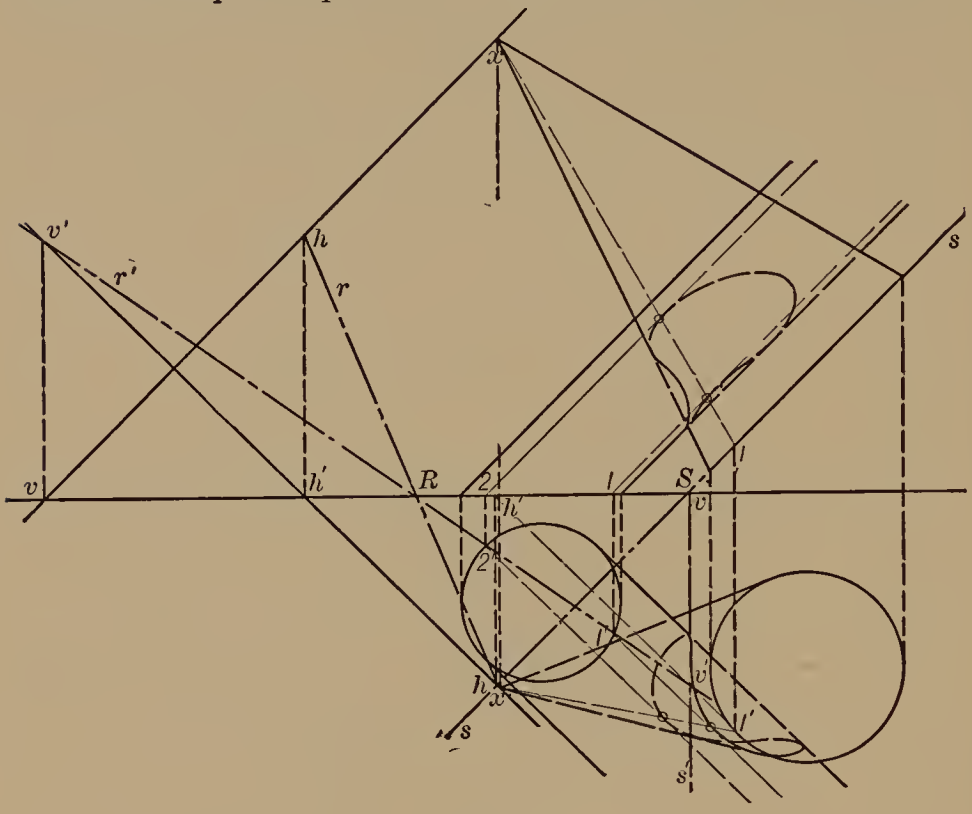


Fig. 9.

One auxiliary plane through OX is T; its H trace is *Tt*. Plane T intersects cone X in elements 1 and 2 and cone O in elements 1 and 2; elements 1 and 2 of cone O intersect elements 1 and 2 of cone X in four points of the required line of intersection; two of the four points lie on the larger curve, the other two on the smaller curve.

SPECIAL CASES

(123) In certain cases the solution is somewhat more involved than those shown above, and other methods may

be necessary. If the base of either surface does not lie in either plane of projection, methods may be used similar to those illustrated in Figs. 8, 9, and 10.

In Fig. 8 the cone is extended until the base lies in a plane of projection.

In Fig. 9 the intersection of each auxiliary plane with the plane of the base of the cone is used to determine the elements of the cone.

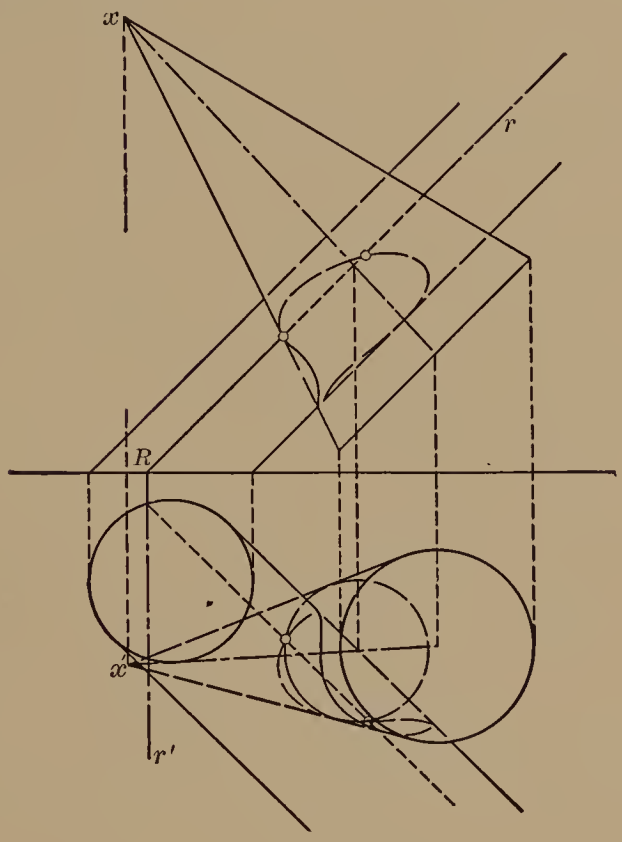
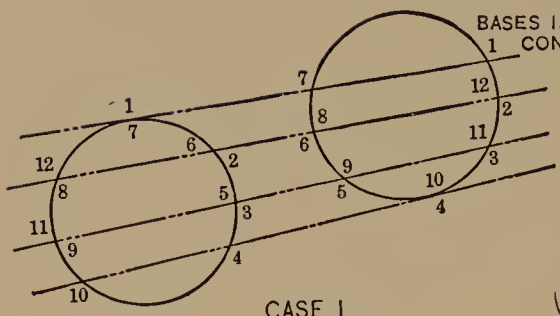


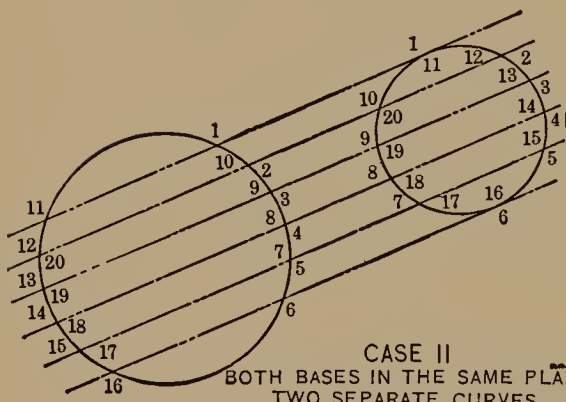
FIG. 10.

In Fig. 10 the auxiliary planes are passed parallel to the base of the cone, cutting from the cone ellipses which project as circles, and from the cylinder elements.

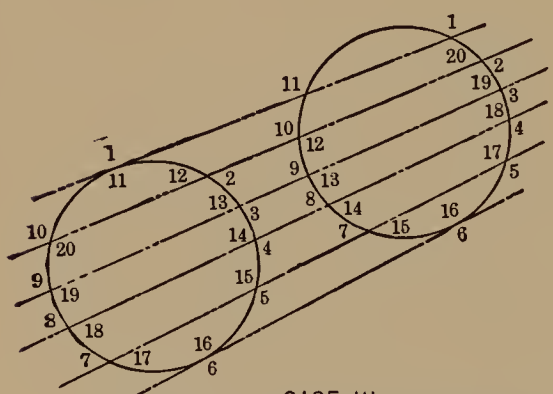
(118) If either surface is a right cylinder with no base shown, it becomes necessary to locate corresponding projections of each element to be used, or to construct a base. For the latter a plane may be passed perpendicular to the



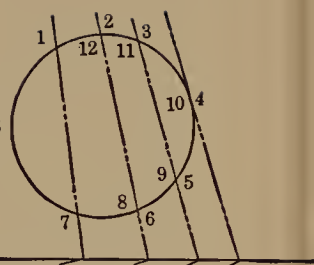
CASE I
BOTH BASES IN THE SAME PLANE
CONTINUOUS CURVE



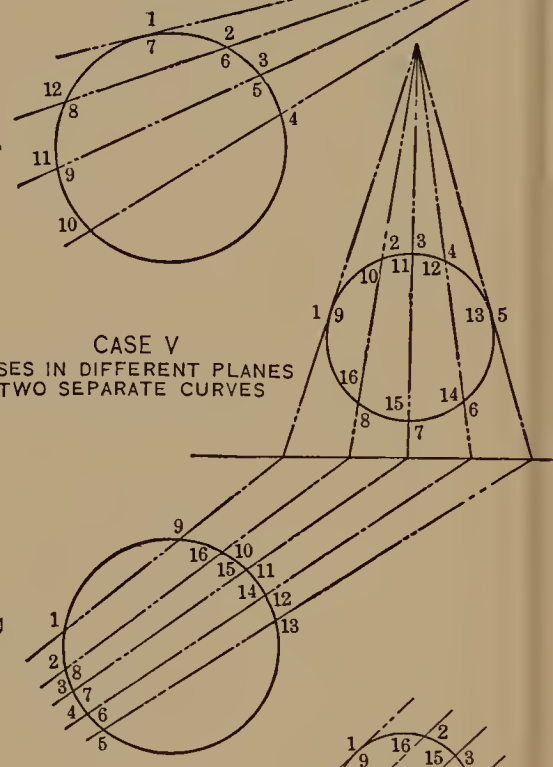
CASE II
BOTH BASES IN THE SAME PLANE
TWO SEPARATE CURVES



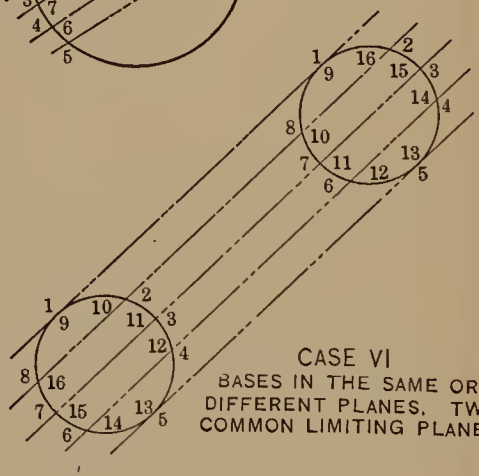
CASE III
BASES IN THE SAME OR DIFFERENT PLANES. ONE COMMON LIMITING PLANE.



CASE IV
BASES IN DIFFERENT PLANES
CONTINUOUS CURVE



CASE V
BASES IN DIFFERENT PLANES
TWO SEPARATE CURVES



CASE VI
BASES IN THE SAME OR DIFFERENT PLANES. TWO COMMON LIMITING PLANES.

FIG. 1.

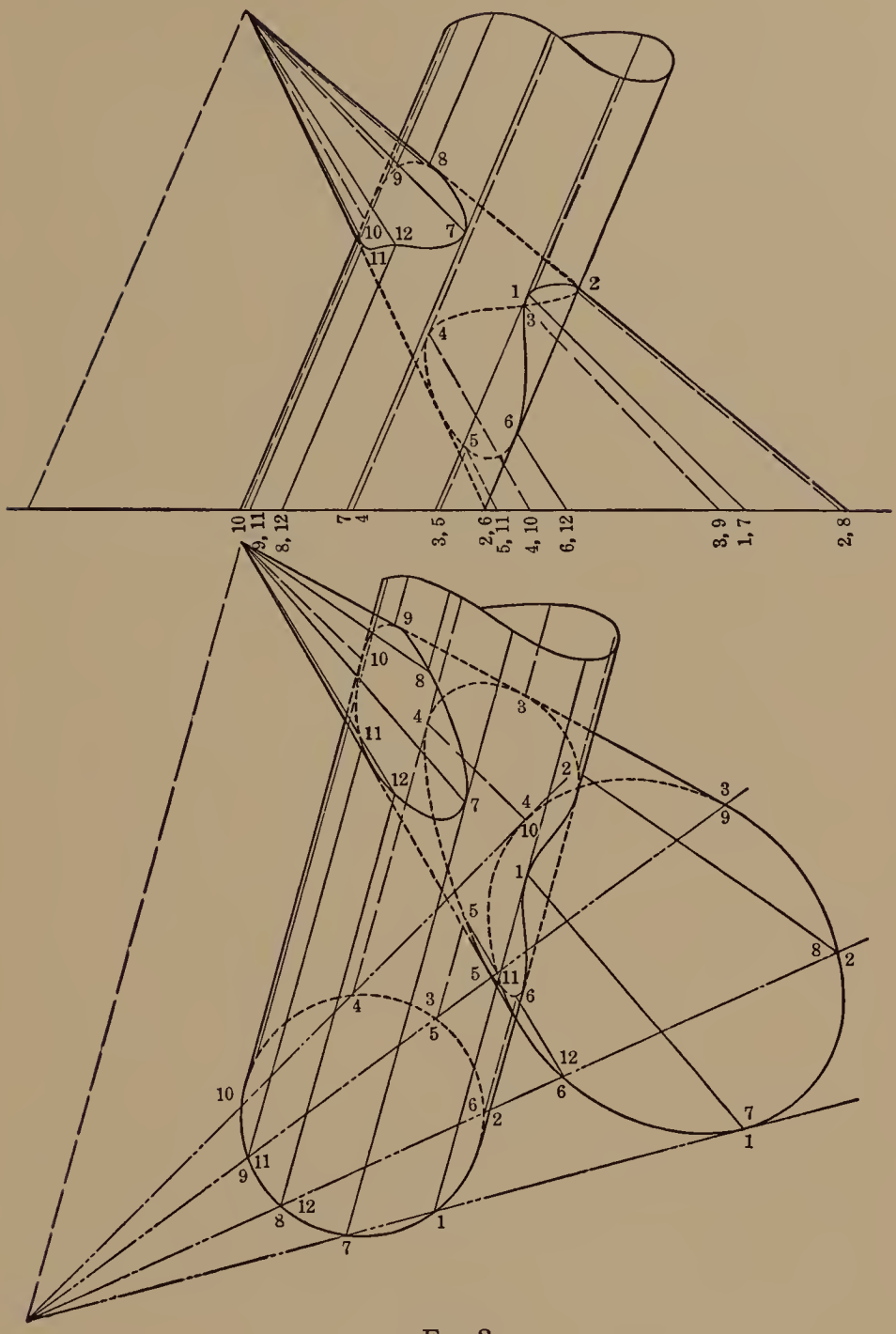


FIG. 2.

axis, intersecting the surface in a right section, which by hypothesis is a circle. If this circle be revolved into or parallel to one coordinate plane, any number of points may be located on it. DE, Fig. 11, is the axis of such a cylinder. Here the cylinder is placed in a position parallel to V, and the circle may be revolved about the diameter 1–7 which is parallel to the V plane until it is parallel to V. The point 3 whose V projection is 3′ is revolved to 3_r and the distance

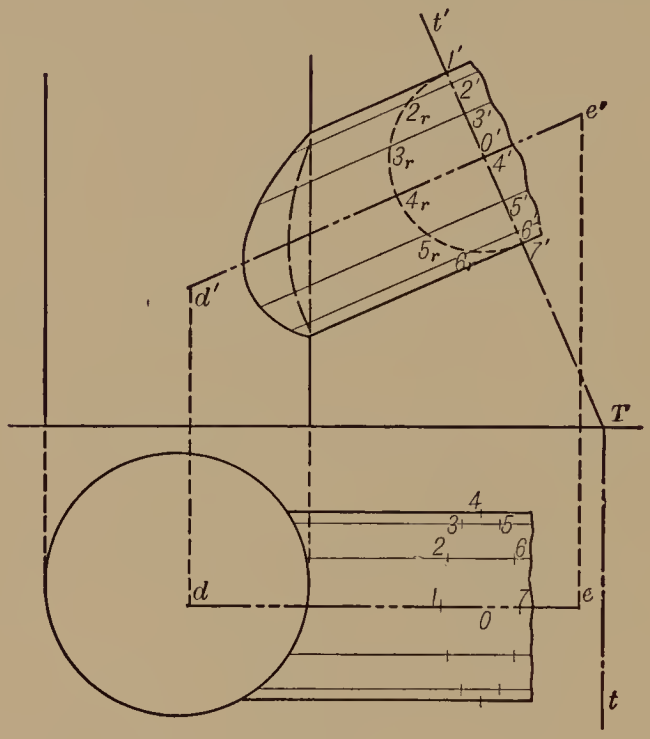


Fig. 11.

from 3 to the diameter 1–7 is the length of the line seen in revolved position as $3_r3'$. This line in its true position is parallel to H and perpendicular to 1–7. The H projection of 3 is therefore this same distance from the H projection of the axis DE.

CHAPTER IX

DEVELOPMENT OF CYLINDERS AND CONES

(125) The development of a cylinder or cone is the surface of that body rolled out on a plane; that is, a plane is passed tangent to the cylinder or cone through one of its elements and the surface is then split along any element and rolled out on this tangent plane. A funnel which is in reality a cone can thus be made from a flat sheet of metal.

In many cases, the surface to be developed may be most easily and rapidly treated by dividing it into two or more identical parts. When one of these has been developed, the entire development may be determined by using properly this first part as many times as it occurs on the given surface. For example, if an oblique circular cylinder with parallel bases is to be developed, only one quarter need be worked out according to the theory given later; the whole development is then obtained by properly using this one part four times.

(126) Problem 35.—**To develop the surface of any cone, and trace upon the development the line of intersection of the cone and any surface.**

Discussion:

If within the given cone a pyramid of a sufficiently large number of sides be inscribed the development of this pyramid may, for all practical purposes, be considered the development of the cone. As the pyramid is composed of a number of triangular faces, its development consists of a number of triangles joined together with vertices at a common point and with sides equal to the true lengths of the edges of the inscribed pyramid. If a smooth curve be

traced through the extremities of these edges, the development becomes approximately that of the cone.

Solution:

If a series of elements of the cone be assumed and considered the edges of an inscribed pyramid, the development of this pyramid with a curve substituted for the straight-line bases is approximately the development of the cone. If from the vertex the true distances from the apex to the different points of the curve of intersection be measured on the proper elements of this development, the curve

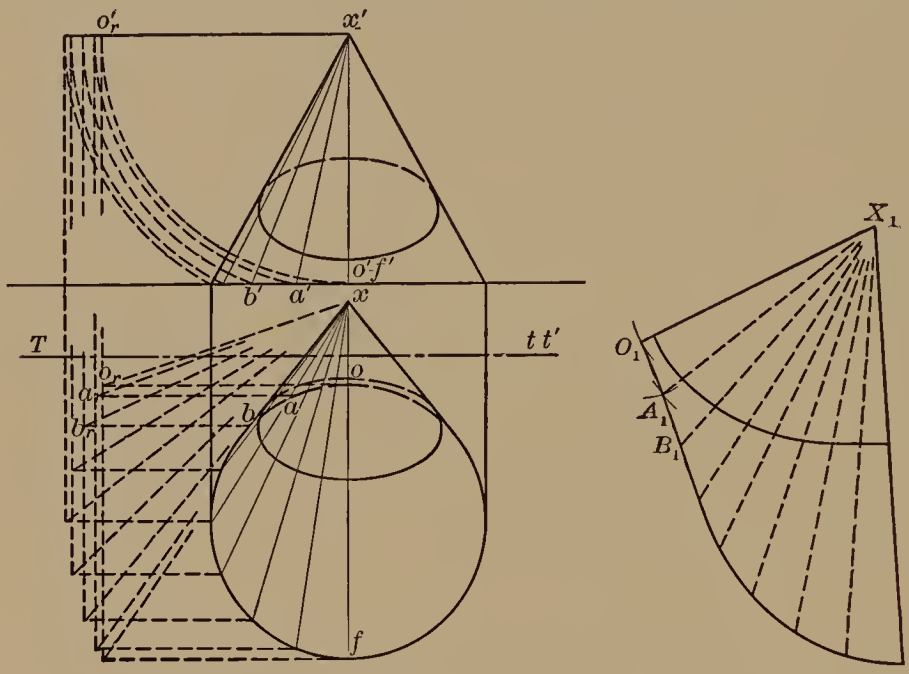


Fig. 1.

traced through these points is the development of the curve of intersection.

Construction:

Given: Cone X and its line of intersection with plane T, Fig. 1.

Reqd: Development of cone showing the line of intersection of cone and plane T.

The base is divided into any number of equal parts, for example 16, as shown, and elements are drawn through

the points of division. As the surface of this cone may be divided into two identical halves, the development of one-half is sufficient. One of these halves has for its base the semicircle OBF. X_1 is assumed in any convenient position on the drawing as the common vertex of the developments of the triangular faces. The line X_1O_1 is drawn from X_1 equal to the true length xo_r of element XO. With X_1 as a center and a radius equal to the true length xo_r of element XA, a short arc is described; then with O_1 as a center and a radius equal to the chord oa, a second arc is described intersecting the first at A_1. This triangle $X_1O_1A_1$ is the development of the triangular face XOA. So in succession the triangular faces $X_1A_1B_1$, etc., are added until the entire half surface is developed. The development of the line of intersection is determined by laying off on the proper elements the true distances from the apex to the several points on the line.

(127) Problem 35: Example 1.—**To develop a right circular cone and trace upon the development the curve of intersection of the cone and any surface.**

Discussion:

This is a special case of the preceding problem. In applying the same method here the base becomes a circle, inasmuch as the elements are equal.

Solution:

If with a radius equal to the true length of an element of the cone, an arc be drawn equal in length to the circumference of the base, this arc and the radii from its extremities enclose the development of the cone.

Construction:

Given: Cone X and its line of intersection with plane T, Fig. 2.

Reqd: Development of cone showing line of intersection of cone and plane T.

Twelve elements are assumed; any number may be used. Any point x_1 is assumed as the development of the

apex of the cone; with x as a center and with a radius equal to the true length of the elements, an arc is described whose total length is determined by stepping off on it twelve arcs corresponding in length to the arcs of the base of the cone. (If the base of the cone is not parallel to H or V the true size must be found.) The division points of these arcs are numbered in order as shown. Points 11 and 11 are joined with x_1 completing the development of the cone.

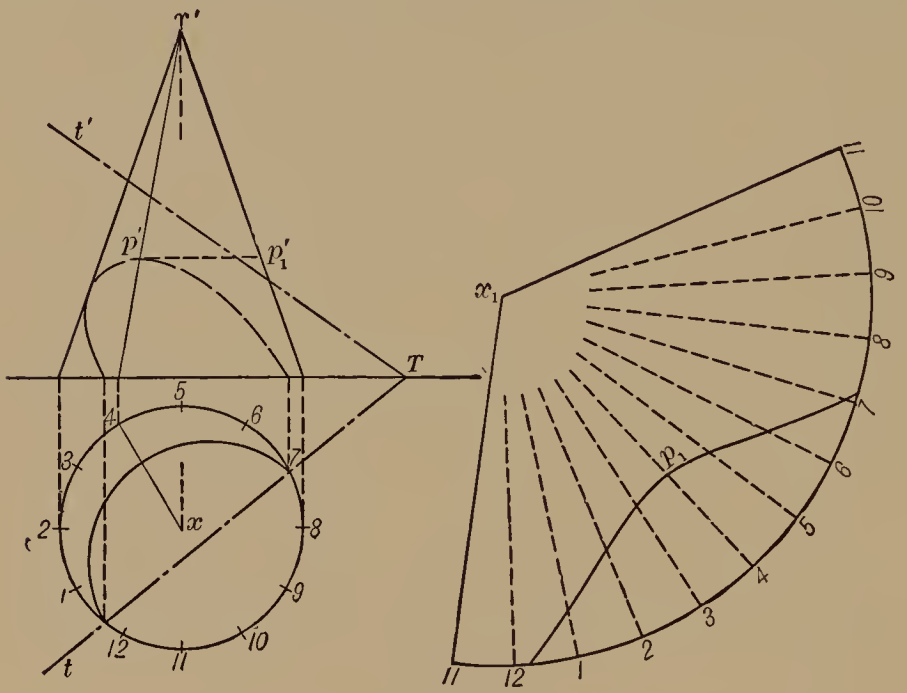


FIG. 2.

The line of intersection is determined by laying off on the development of the proper elements the *true distances* from the apex to the respective points on the line.

The true length of the portions of element 6 from X to P may be determined by revolving the cone about its axis until that element is parallel to V, when its V projection will coincide with one side of the triangular V projection of the cone, and shows in its true length. To find the length of XP, then a line parallel to GL is drawn through p' until it intersects the side of the triangle at p'_1; $x'p'_1$ is the required true length.

(128) Problem 36.—**To develop any cylinder and trace upon the development the line of intersection of the cylinder and any surface.**

Discussion:

If a plane be passed perpendicular to the elements of the cylinder, it will intersect the cylinder in a right section which will develop as a right line, with the elements per-

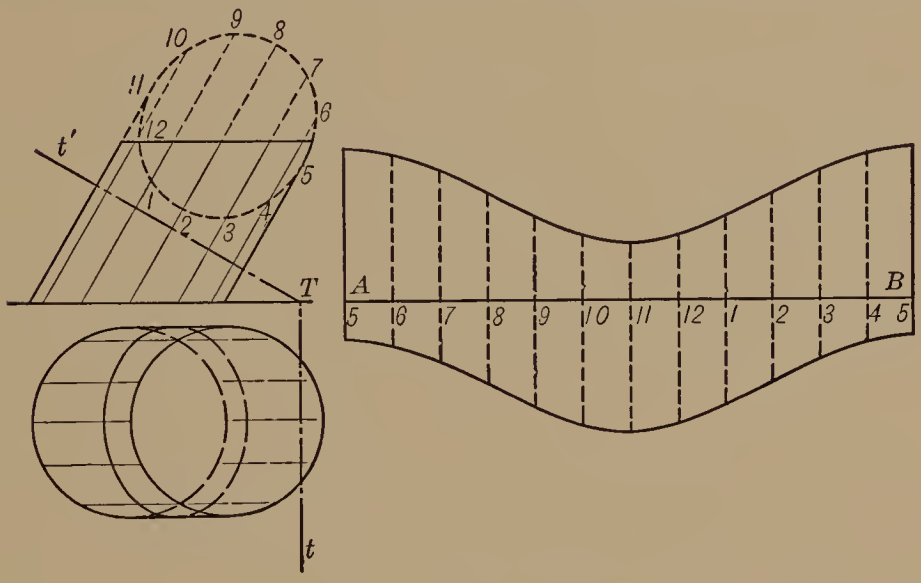


Fig. 3.

pendicular to it. Points on the cylinder on either side of this right section will appear in development on their respective elements and at the proper distances on the one side or the other of this right line.

Solution:

If a plane be passed perpendicular to the axis of the cylinder it intersects the cylinder in a right section whose development is a right line perpendicular to the developments of the elements. If the true distances from the intersections of elements with the right section to the respective bases and lines of intersection be laid off on the developments of these same elements, the curves traced through the points so determined are the developments of the bases and lines of intersection.

Construction:

Given: The oblique cylinder Fig. 3, and line of intersection with plane T.

Reqd: Development of cylinder, showing line of intersection of cylinder and plane T.

Plane T is passed perpendicular to the elements of the cylinder; it intersects the cylinder in the right section shown. Plane T is then revolved about Tt' into V, where the revolved right section appears in its true size. Any

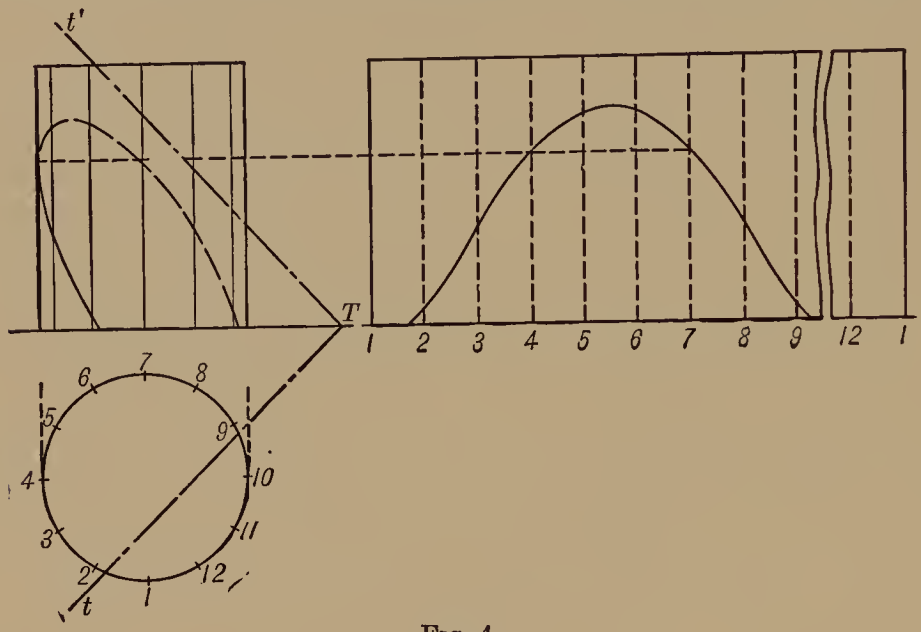


Fig. 4.

number of elements may be assumed, e.g., 12; the true distances between these elements are shown in the revolved position of the right section. The development of the right section is the line AB. The several points of this line, 5, 6, 7, etc., are determined by laying off distances equal to the rectified arcs of the revolved right section, and through these points the elements are drawn perpendicular to AB. The lengths of these elements on either side of AB are determined by the true distances from the right section to the bases of the cylinder. Curves drawn through the extremities of the elements complete the development of the

cylinder. Any line of intersection on the surface is located in the same manner as were the curves for the bases; i.e., by finding the distances from the right section to a series of points on the curve.

The construction for the cylinder whose axis is perpendicular to H is shown in Fig. 4.

Should the cylinder be placed in a position other than parallel to V, Fig. 5, a new V plane may be drawn parallel

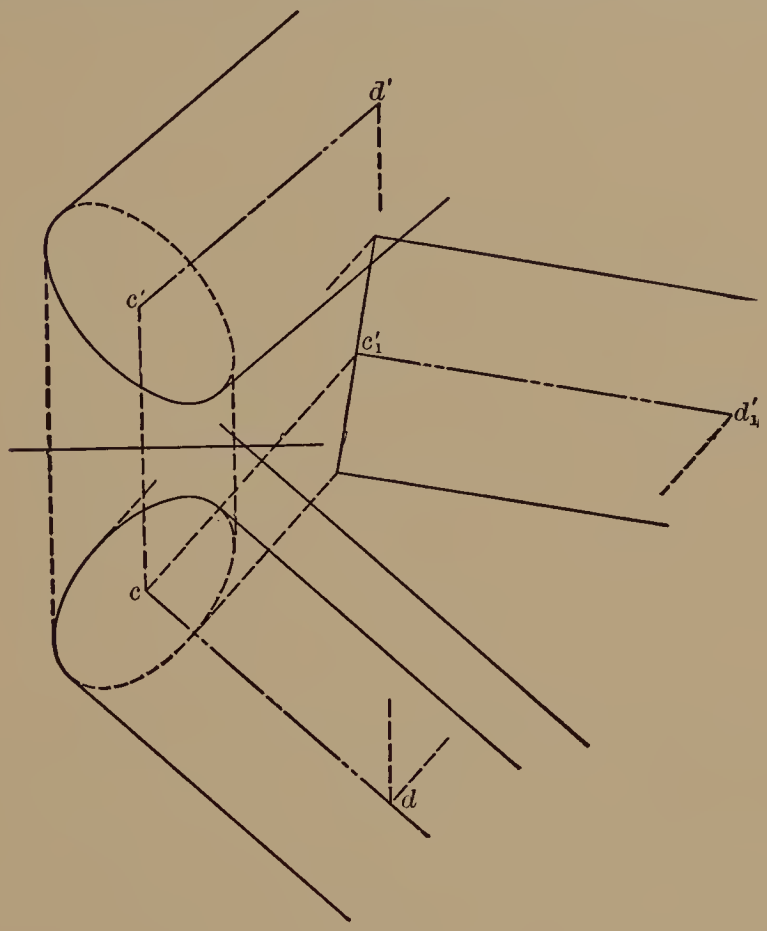


FIG. 5.

to the elements of the cylinder. After the projection of the cylinder is constructed on this new V plane, the development of the surface may be obtained as in Fig. 4. By this method all of the elements are shown in their true length in this auxiliary projection.

PART III

CHAPTER X

SHADES AND SHADOWS

SCOPE

(129) In work of an architectural nature, quite frequently for the sake of natural effects, it is desirable to show the shades and shadows cast by the different parts of an object. However, it is not the intention of this text to treat the subject from any but a strictly mathematical standpoint, leaving the more elaborate discussions of degrees of shade, etc., to other works. Likewise only such theory will be given as seems to be necessary for ordinary practical purposes.

LIGHT—SOURCE—RAYS

(130) Shades and Shadows, indicating the absence of a certain amount of light from some surface, of course, always presume the *existence* and a *source* of light of some sort. Ordinarily, for outdoor work this source is the *sun;* and for the easy treatment of the subject in physics, etc., a shaft or beam of sunlight is imagined to be composed of countless *parallel rays,* these rays being merely *mathematical lines.*

DIRECTION OF LIGHT

(131) Upon the assumption of a source of light and the existence of light rays, the *direction* of these rays must be

either determined or assumed. Shadows being used chiefly for effects, there can be no objection to the assumption of the direction of the rays. However, in assuming this direction, some care should be taken to make it approximately what it actually is for that *latitude*. A reasonably good rule is to assume both projections of the light rays to make with GL an angle equal to the *complement* of the *latitude* of that section of the country. The angle commonly used in the northern half of the United States is 45°. As it has been found best to work with the light coming from the

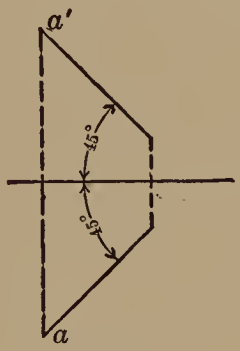


FIG. 1.

left, the projections of the rays will slope *toward* GL to the *right*, when objects are placed in the 1st quadrant, Fig. 1.

DIRECT LIGHT—INDIRECT LIGHT

(132) All light received from the source is known as **direct light,** and any portion of any surface from which the *direct light* is excluded by some body is said to be **in shadow.** However, it must not be imagined that these shaded surfaces receive no light at all, for in some cases it may be almost impossible to distinguish with the eye between the *shaded* and *illuminated* portions of a surface. There are always countless surfaces which reflect the direct light upon the shaded surfaces, possibly lighting them more brilliantly than they could be by direct light. This *reflected* light is known as **indirect light** and gives rise to *degrees* of *shade* mentioned before.

SHADES—SHADOWS

(133) Any portion of a surface of an object from which the *direct* light is excluded by some part of the *same* object is said to be **in shade**, or for convenience may be called a **shade.**

Any portion of a plane or the surface of an object from which the **direct** light is excluded by some part of **another** object is said to be **in shadow**, and for convenience may be called a **shadow.**

LINE OF SHADE—SHADOW

(134) The line which *separates* the shaded from the illuminated portions of any surface is known as the **line of**

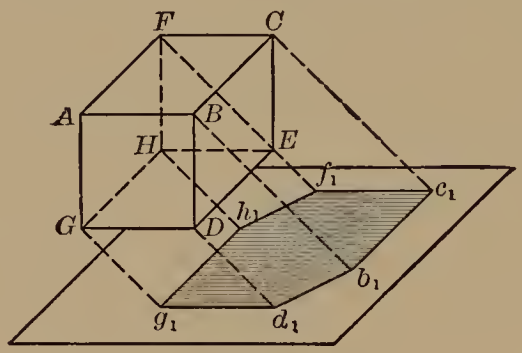


Fig. 2.

shade; and the line which *limits a shadow* on any surface is called a **line of shadow.** See c_1b_1, Fig. 2.

SHADOW RAY

(135) In finding the shadow of an object on a surface, considerable labor can be saved if one determines by inspection what lines of the object will determine the limits of the shadow, e.g., in Fig. 2 it is easily seen that point A will play no part whatever in determining the shadow, while points E, H, F, C, B, G, and D will. It is seen that by passing a ray of light through B, the point b_1 is located; likewise, c_1, etc. These are significant points in the line of

shadow and when located carefully may save much labor. The line Bb_1 is in reality a ray of light; however, inasmuch as it has been used in determining a point on the *line of shadow* it will be known in this work as a **shadow ray.** Although the term may be absurd, however, its value as a working tool cannot be doubted. Any other such ray which may be used to *determine* any *point* either on a *line* of *shade* or a *line* of *shadow* will be known as a **shadow ray.**

FIRST VERSUS THIRD QUADRANT

(136) Shadows are rendered in drawings of one view only. Ordinarily this view is an elevation. Since the top view, or plan, does not have the shadows shown, it is advantageous to place the object in the *first*, rather than in the *third*, quadrant. By this arrangement the V plane may easily be conceived to be a wall of any building upon which shadows may be cast. Further, the student seems to find it more natural to imagine the H plane to be the ground or plane of the base of the object than to use the extra plane when the object is in the third quadrant.

SHADES AND SHADOWS—HOW REPRESENTED

(137) It is easily understood that any given point can have only *one* shade or *one* shadow; likewise, that an object can be viewed from only *one* point at one time. Hence, when objects are viewed from *before* V, i.e., lines of sight are perpendicular to V, as far as the observer is concerned there are *no* shades nor shadows on *horizontal* planes, and unless so specified in the requirements of the problem such shades and shadows need not be found.

Concerning shades and shadows on *oblique* planes, even though this whole subject is here kept within the realm of orthographic projection, it is seen that when observed from *before* V the shade or shadow of a given point on a given *oblique* plane is represented by the V *projection* of that shade or shadow; as far as the observer is concerned an H *pro-*

jection does not exist and need not be found. When viewed from above H the H *projection* of the shade or shadow represents that shade or shadow and the V projection does not exist and need not be found.

(138) Problem 37.—**To find the shadow of a given point on a given surface.**

Solution:

If through the given point a shadow ray be passed, the point in which it pierces the surface is the required shadow.

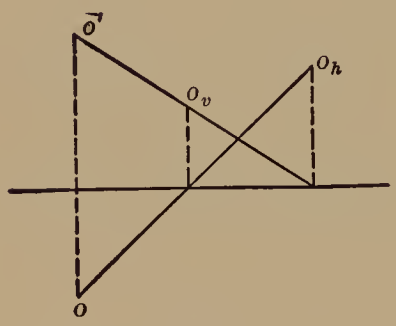


Fig. 3.

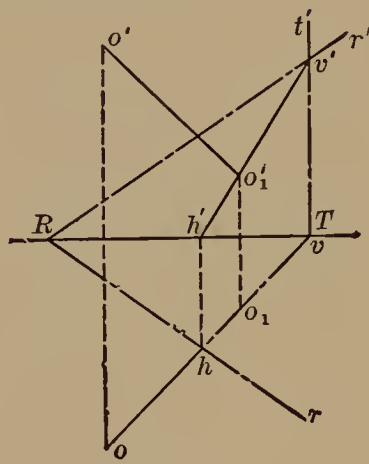


Fig. 4.

Construction: (*a*)

Given: Point O, Fig. 3.

Reqd: Shadow of O on V or on H.

The projections of the shadow ray are drawn through o' and o at 45° to GL; this ray pierces V at o_v the required shadow on V, or H at o_h the shadow on H.

Construction: (*b*)

Given: Point O and plane R, Fig. 4.

Reqd: Shadow of O on R.

An H projecting plane is passed through the shadow ray from O; its line of intersection with R is VH. The shadow ray intersects this line at O_1, o_1 and o'_1. If shadows are required in elevation, o'_1 is the desired shadow; in plan, o_1 is used.

Construction: (*c*)

Given: Point O and cone X, Fig. 5.

Reqd: Shadow of O on surface of cone.

The shadow ray is passed through O, and O is connected with the apex X. Plane T of these two lines cuts the element CX from the cone; the shadow ray intersects this

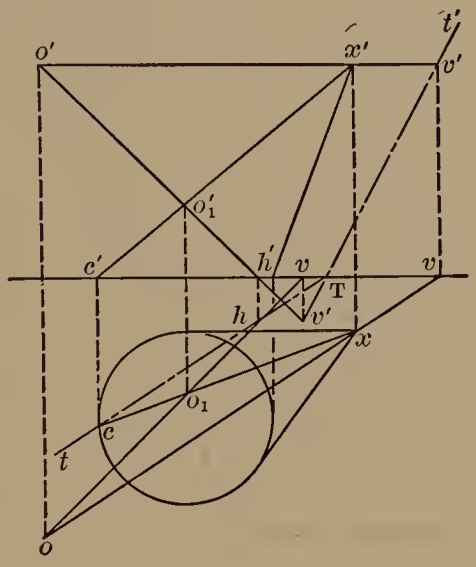


Fig. 5.

element in O_1. In dealing with shadows in elevation, o'_1 is the required shadow; in plan, o_1 is used.

Construction: (*d*)

Given: Point O and the ellipsoid X, Fig. 6.

Reqd: Shadow of O on ellipsoid.

The H projecting plane T is passed through OC; it cuts from the ellipsoid an ellipse whose minor axis is *aa*. The major axis *d'd'* is obtained by laying off the distance *e* on the V projection as shown. Plane T is revolved parallel to V about the axis of the ellipsoid as an axis of revolution. OC comes into the position O_rC_r, and the V projection of the revolved position of the ellipse may be constructed in the position shown, with minor axis equal to *aa* and major axis equal to *d'd'*; the ordinary methods of constructing an ellipse may be used. The revolved position of the required shadow is

then found at o'_1, and the shadows themselves at o'_1 and o_1 respectively.

BRILLIANT POINTS—LINES

(139) If on a surface of double curvature there is a *point* at which a ray of *direct* light is *reflected* to the eye, this point appears more brilliant than any other point of the illuminated surface and is known as a **brilliant point.** On a surface of *single* curvature instead of a point there will be a

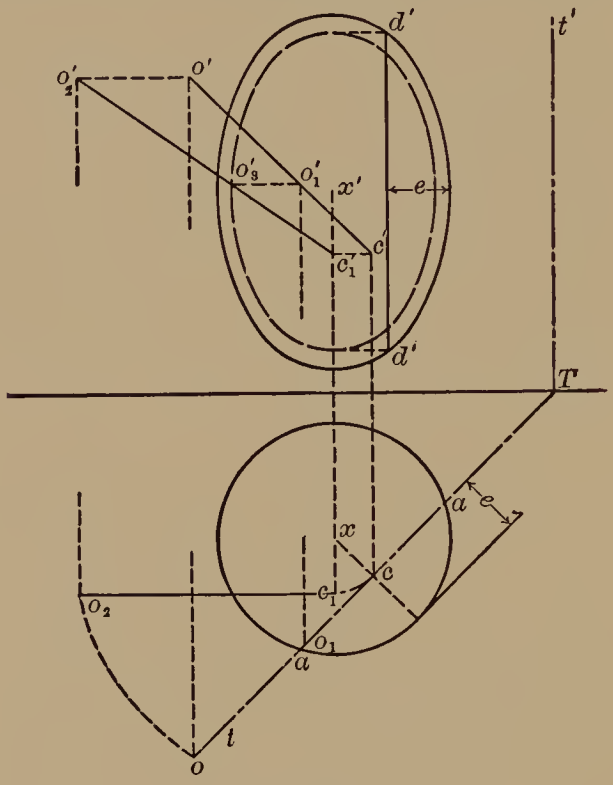


Fig. 6.

line along which *direct* light will be reflected to the eye (if reflected at all); this line is known as the **brilliant line.**

It can be readily seen that such *brilliant points* or *lines* do not necessarily exist for every position of the point of sight and any assumed direction of light; however, if the surface be either a sphere or an ellipsoid of revolution, such a brilliant point does exist and can readily be found.

(140) Problem 38.—**To find the brilliant point on the surface of a given ellipsoid of revolution.**

Discussion:

Before proceeding with the construction it is necessary to fix well in mind a number of facts of the problem.

1. Assuming that it is desired to locate the brilliant point in *elevation*, the eye is an infinite distance before the V plane; hence the *reflected* ray is *perpendicular* to V. 2. The *bisector* of the *angle* between the *incident* and *reflected* rays is *normal* to the *surface* of the ellipsoid at the *brilliant point*. 3. A *normal* to the *surface* of an *ellipsoid* passes through the *axis* of *revolution*.

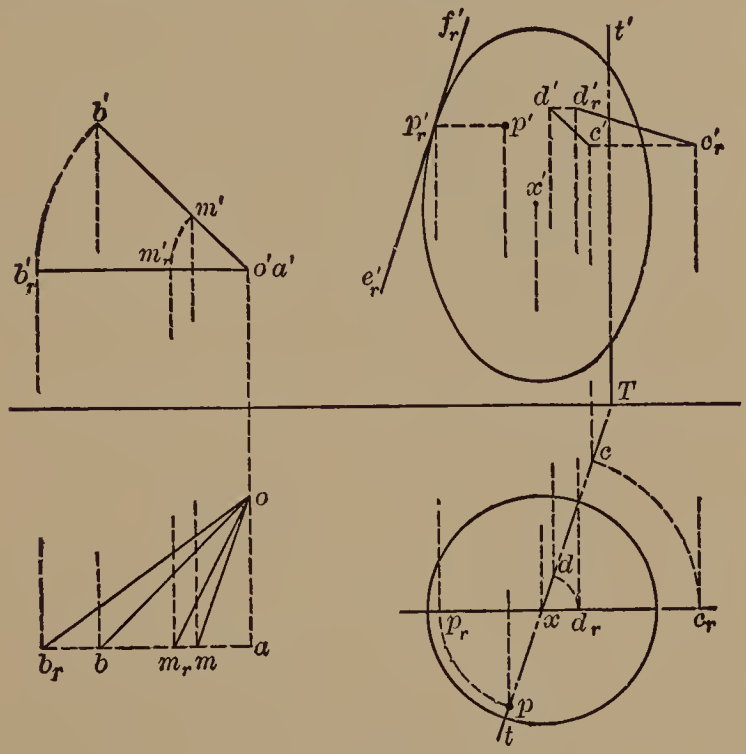


Fig. 7.

Solution:

If through any convenient point in space a ray of light and a line of sight be passed (the latter perpendicular to V), the bisector of the angle between these two right lines is parallel to the normal of the surface at the brilliant point. If then a plane be passed through the axis of the ellipsoid parallel to the bisector of this angle

and a line be drawn in this plane perpendicular to the direction of the normal (or the bisector), and tangent to the ellipse of intersection of the plane with the ellipsoid, the point of tangency of this line with the ellipse is the required brilliant point.

Construction:

Given: The ellipsoid X, Fig. 7.

Reqd: The brilliant point in elevation.

The ray of light OB and the line of sight OA are passed through any point O. Revolving OB parallel to H about OA as an axis, the bisector, om_r is located, thence om and $o'm'$

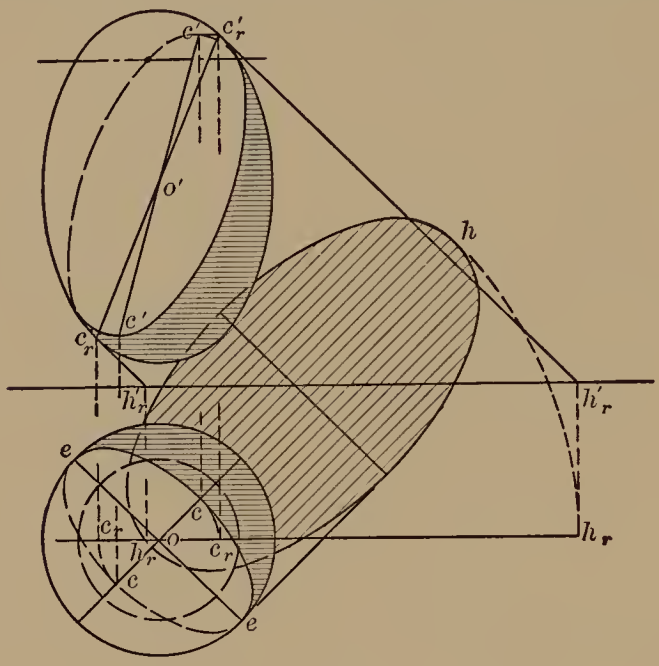


Fig. 8.

the projections of the bisector. Plane T is passed through the axis of the ellipsoid parallel to OM; Tt is parallel to om. The line CD is passed through any point C in T parallel to OM; revolving T about the axis of the ellipsoid as an axis until it is parallel to V, the V projection of the ellipse cut from the ellipsoid by T now coincides with the V projection of the ellipsoid; CD revolves into the position $c'_r d'_r$, $c_r d_r$. The tangent line $e'_r f'_r$, perpendicular to $c'_r d'_r$ locates the V pro-

jection of the revolved position of the brilliant point at p'_r and the required brilliant point is found at p', p.

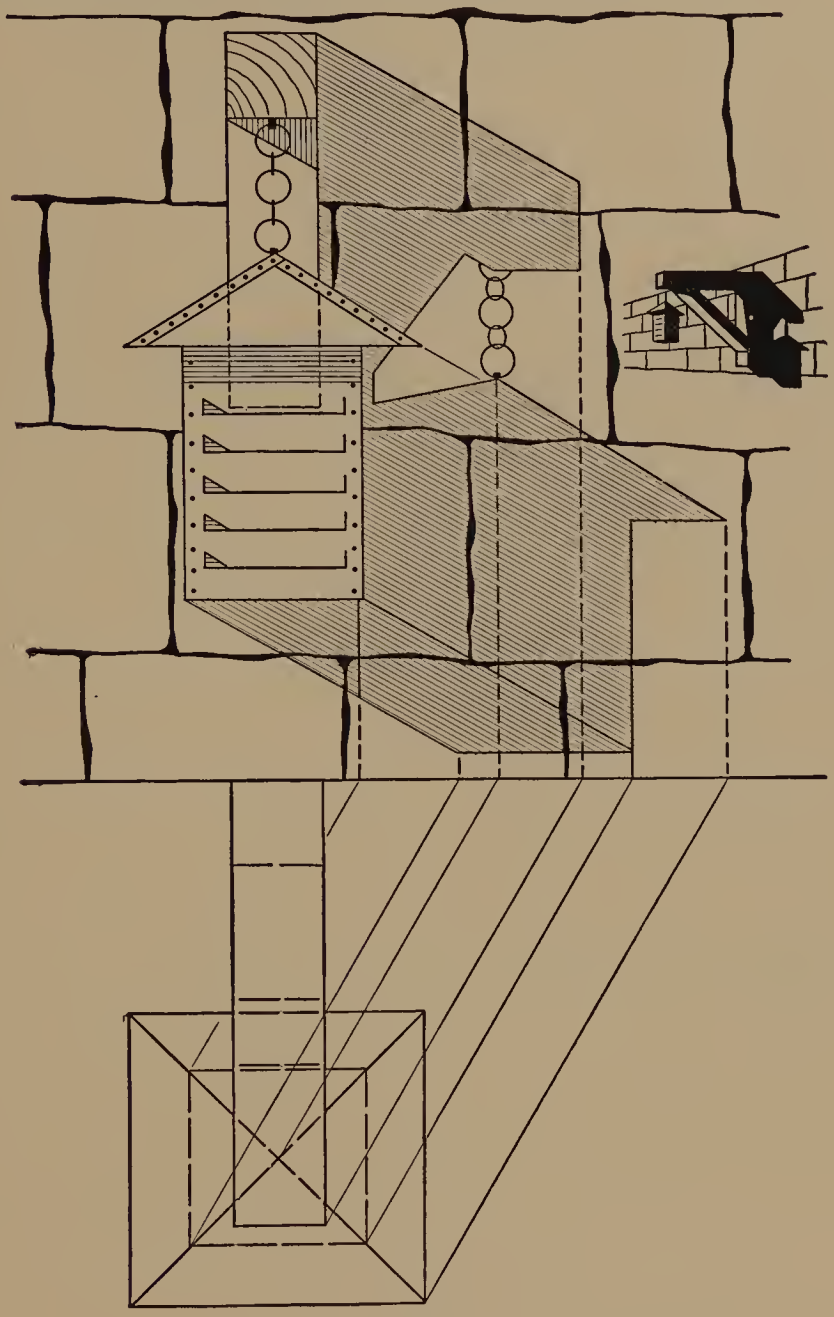


FIG. 9.

(141) Problem 39.—**To find the shade and shadow of a given ellipsoid of revolution on H.**

Solution:

If a cylinder of light rays be drawn tangent to the given ellipsoid, the line of tangency is the line of shade; the line of intersection with H of this cylinder is the required line of shadow.

Construction:

Given: Ellipsoid O, Fig. 8.

Reqd: Shade and shadow of ellipsoid on H.

If the ellipsoid and the cylinder of tangent rays be revolved about the axis of the ellipsoid until the elements of the cylinder are parallel to V, the upper and lower limiting elements come into the positions C_rH_r, projections of which are $c'_rh'_r$, $c'_rh'_r$ and c'_rh_r, c_rh_r. The axes of the elliptical line of shadow on H are determined as equal to $h'_rh'_r$ (major) and the diameter of the circular H projection of the ellipsoid (minor); the line of shadow is then determined as shown. CC is the major axis of the elliptical line of shade; however, its H projection c_rc_r becomes the minor axis of the elliptical H projection of the line of shade; c_rc_r revolves back into the position cc; the chord ee perpendicular to cc at its center is the major axis. c'_r and c'_r revolve back into the positions c' and c' giving two points on the V projection of the line of shade. Other points may be determined by cutting circles from the ellipsoid by auxiliary planes perpendicular to the axis, and finding the points of intersection of these circles with the H projection of the line of shade.

CHAPTER XI

PERSPECTIVE

DEFINITION

(142) A **perspective drawing** of any object is such a **representation** as will have the same appearance as the object itself when viewed from some *definite point.* An object can be viewed from an infinite number of positions, each view differing slightly from the others; hence, any number of perspectives can be made of the same object.

But little use is made of perspective except in architectural work. Working drawings, composed of several views, and made with a corresponding number of viewpoints at infinite distances from the coordinate planes, of course represent little but the bare details of construction. Isometric and oblique projections, though representing an object by a single view, serve little but to convey a conception of the relation of the several parts of an object to each other. Despite all care and allowances such projections never appear quite natural.

It is then left entirely to perspective to represent objects as they actually appear to us. It is true, as mentioned before, that the chief use of perspective is in architectural work; however, the principles of construction, depending upon the principles of Descriptive Geometry, are easily acquired and can be applied in a mechanical way by the engineer without any special artistic training.

PICTURE PLANES

(143) Perspective drawings are ordinarily made from working drawings, and as working drawings are 3rd quad-

rant projections we will assume that always in making a perspective the object is placed in the 3rd quadrant.

Any *plane parallel* to either the V plane or the H plane may be used as the **picture plane**; in this treatise, however, for simplicity of construction, the V and H planes will be termed and used as the picture planes; i.e., the planes on which the *representations* or *perspective drawings* are *made.*

When the representation or drawing is made on the H plane it is termed a **bird's eye view**; such drawings are made frequently of large areas of ground, cities, groups of buildings, etc. If, however, the drawing is made on the V plane it is termed a **perspective**; such drawings of course have the wider range of utility.

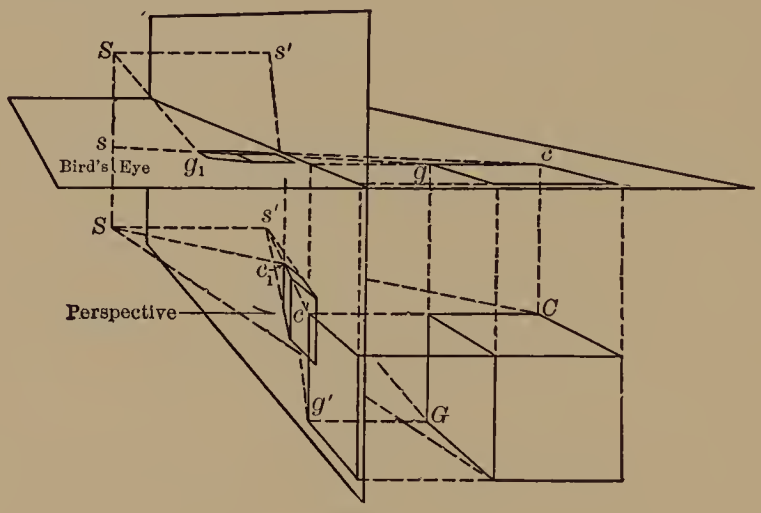


Fig. 1.

POINT OF SIGHT—PRINCIPLE POINT

(144) The *position* of the *eye* of the *observer* is known as the **point of sight** and is lettered S, Fig. 1. This point of sight is on the *opposite* side of the *picture plane* from the *object,* that the drawing may be *smaller* than the object; however, if it is desired to make the drawing larger than the object, a new picture plane may be drawn in such position as to bring both the object and the point of sight on the same side.

In making a *bird's-eye view* the *point* of *sight* will be *above* the H plane and, ordinarily in the 1st quadrant; while for a perspective, S is taken in *front* of V and usually in the 4th quadrant.

The V projection of S will be lettered s', the H projection s. s' is of great importance in construction, hence is known as the **principal point,** its use will be explained later.

There is more to the locating of the point of sight with reference to the object than appears on first thought. The principles of construction can be applied as easily with S in any one position as in any other; however, unless care is exercised, the drawings, though mathematically correct, may have a most distorted appearance, e.g., one can imagine how absurd the perspective of a house would appear when made with the view point only a few feet from one of the walls. Suggestions for placing the point of sight will be given later.

VISUAL RAY—PLANE—CONE

(145) A *line* from the *point* of *sight* to any *point* of an *object* is known as a **visual ray.** A *plane* passing through the *point* of *sight* and any *right line* of the *object* is a **visual plane.** A *cone* of *visual rays* with the *apex* at the *point* of *sight* and the *surface* of the cone *tangent* to a *given curved surface* (sphere, etc.), or all *rays intersecting* a *given curved line* is known as a **visual cone.**

PERSPECTIVE BY PIERCING POINTS

(146) Problem 40.—**To find the perspective of a given point.**

Solution:

The piercing point in V (the picture plane) of a visual ray through the point of sight and the given point is the required perspective.

Construction:

Given : Point O, 3rd quadrant, Fig. 2.

Reqd: Perspective of O.

$s'o'$ and so are the projections of the visual ray SO. This ray pierces V at V, projections v' and v. The point v' is the required perspective and will be lettered o_1 if lettered at all.

Defect: The theory of the problem just given is sufficient to construct completely the perspective of any object; however, in practical work, with few exceptions the method cannot be used; for the point of sight is invariably

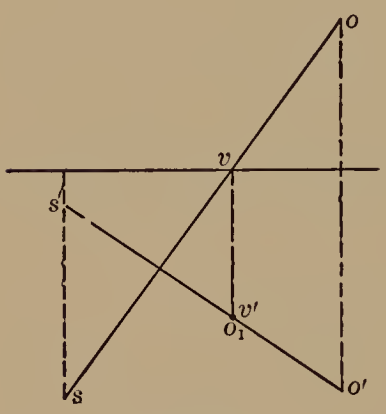


Fig. 2.

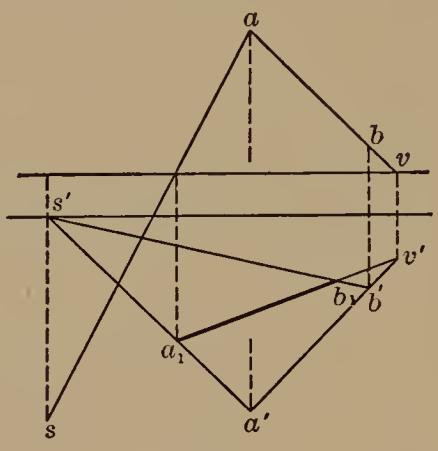


Fig. 3.

in such position as to bring its H projection s without the limits of the drawing. This theory is, however, the foundation of problems to follows.

(147) Problem 41.—**To find the perspective of a given line.**

Discussion:

From the previous problem it is seen that if a point lies in V, the piercing point of the visual ray and the given point coincide; i.e., the *point* is its *own perspective.* Then if a given line be produced until it pierces V, this point is one point of the perspective of the given line.

Solution: (*a*)

If the given line be extended until it pierces V (the picture plane), and a visual ray be passed through the point of sight and some point of the given line and extended until it pierces the V plane, the line passing through these two piercing points is the required perspective.

Construction:

Give: Line AB, 3rd quadrant, Fig. 3.

Reqd: Perspective of AB.

AB produced pierces V at V which coincides with v' and the visual ray from S to A pierces V at a_1. The perspective of B lies on the V projection of the visual ray from S to B, hence b_1 is at the intersection of $s'b'$ and a_1v'. a_1b_1 is the required perspective.

Solution: (*b*)

The V trace of a visual plane through the point of sight and the given line is the required perspective of the given line.

SPECIAL LINES

From Solution (*b*) above, it is seen that if a *line* is *parallel* to the *picture plane* (V), its *perspective* is *parallel* to the *given line*, likewise to the V *projection* of the given line.

The *perspective* of a *vertical line* is a *vertical line*.

(148) Problem 42.—**To find the vanishing point of a given group of parallel lines.**

Discussion:

If a group of parallel lines be produced an infinite distance they *appear* to *meet* in a point known as their **vanishing point.** A visual ray parallel to this group of lines likewise appears to pass through this vanishing point, hence its piercing point in V is the perspective of this point. In perspective this *piercing point* is known as the **vanishing point** of this group of lines. The perspective of each line of the group passes through the vanishing point.

It is seen from the figure that after the vanishing point of a group of parallel lines is found, it is necessary to find

the perspective of only *one* point of each line to determine the perspective of the whole line; the vanishing point, being common to all of the perspectives, furnishes a second point.

Solution:

The piercing point in V (the picture plane) of a visual ray through the point of sight parallel to any line of the given group is the required vanishing point.

Construction:

Given: Point of sight and lines, AB, CD, and EF, Fig. 4.

Reqd: Vanishing point of given lines.

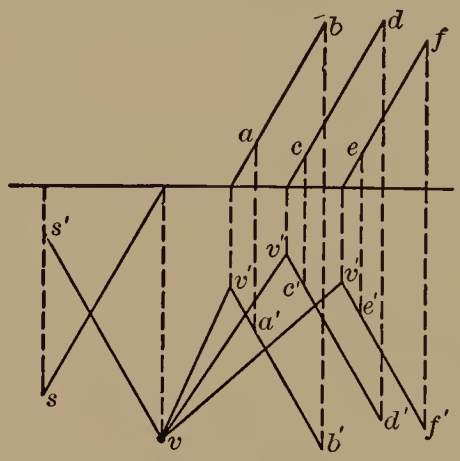


FIG. 4.

The projections of the ray are drawn through s and s' parallel to the respective projections of the given lines. Its piercing point in V (the picture plane) is V, the required vanishing point.

PERPENDICULARS—HORIZONTALS—DIAGONALS—POINTS OF DISTANCE

(149) Lines *perpendicular* to the V *picture plane* are known as **perpendiculars**; such lines are of course horizontal.

If a group of perpendiculars be produced indefinitely they appear to meet in a point on what we commonly call the **horizon.** A visual ray parallel to these perpendiculars

pierces V in s'; hence s' is the *vanishing point* of all *perpendiculars*. A *horizontal* line through s' and lying in the picture plane is the *perspective* of the *horizon* mentioned above; this horizontal line through s' is known in perspective as the **horizon.** All *horizontal lines vanish* in *points* on the *horizon*, Fig. 5.

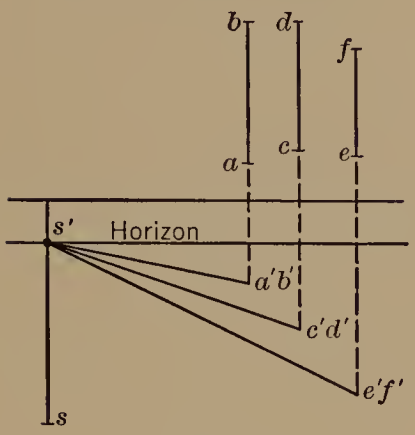


Fig. 5.

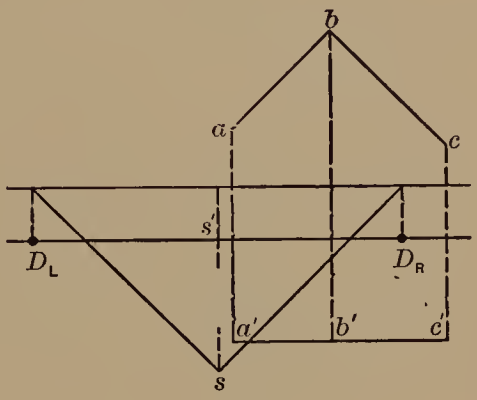


Fig. 6.

Any *horizontal line* making 45° with the *picture plane* is known as a **diagonal**, e.g., AB and BC, Fig. 6. *Diagonals vanish* in *points* of the *horizon*; on inspection of Fig. 6 it will be seen that these vanishing points D_R and D_L of *diagonals* are on the *horizon* at distances to the *right* and *left* of s' equal to the *distance* that s is *below* GL, or that S is *before* the *picture plane.* Hence D_R and D_L, also known as **points of distance**, are readily located whether s is without the limits of the drawing or not.

(150) Problem 43.—**To find the perspective of a given point by diagonals and perpendiculars.**

Discussion:

If through any given point two lines be passed and the perspectives of these two lines be found, the point of intersection of the perspectives is the perspective of the given point.

Solution:

If through the given point a perpendicular and a diagonal be passed the intersection of their perspectives is the perspective of the given point.

Construction:

Given: Point O, 3rd quadrant, Fig. 7.

Reqd: Perspective of O by a perpendicular and a diagonal.

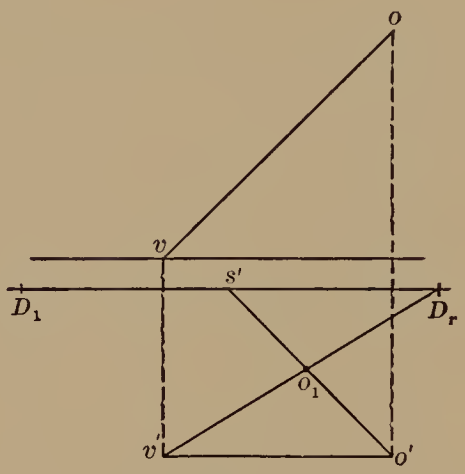


Fig. 7.

D_R and D_L are located at equal distance from s'. The diagonal through O pierces V at v' and its perspective is $v'D_R$. The perpendicular through O pierces V at o' and its perspective is $o's'$, $v'D_R$ and $o's'$ intersect at o_1 the required perspective.

Defect: Though the construction explained above can be, and generally is, used in perspective construction, problems will arise in which even the points D_R and D_L fall without the limits of the drawing. In such cases a geometrical modification of the method is used; i.e., points known as **reduced points** of **distance** are used instead of D_R and D_L; these *reduced points* of *distance* are located on the *horizon* at distances to the *right* and *left* of s' equal to some *simple fractional part* of $s'D_R$, e.g., $\frac{1}{2}$, $\frac{1}{4}$, etc. Such points are lettered D_{2R} and D_{4R}. See Fig. 8.

(151) Problem 44.—**To find the perspective of a given point by perpendiculars and diagonals using reduced points of distance.**

Construction:

Given: Point O, 3rd quadrant, Fig. 8.

Reqd: Perspective of O.

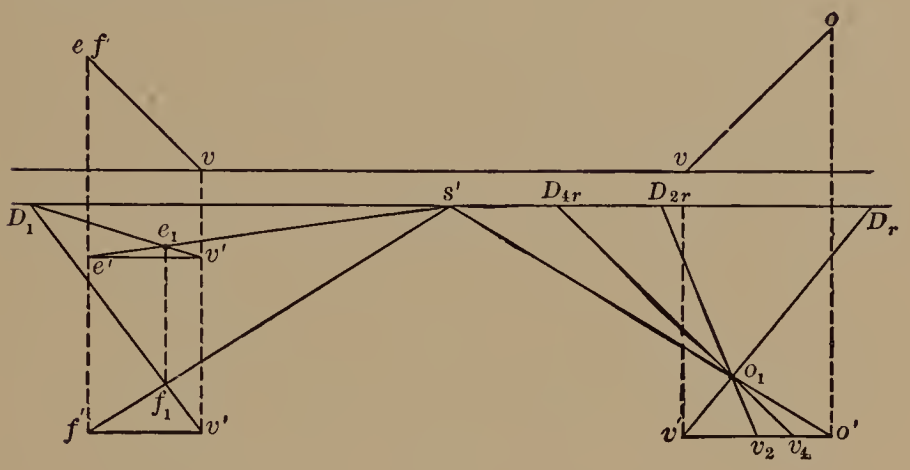


FIG. 8.

The perspective of the perpendicular through O is $o's'$. The V piercing point of the diagonal is at v'. The perspective of the diagonal is $v'D_R$ and the perspective of O is o_1. If D_R is inaccessible and D_{2R} can be used, the line from v_2, midway between o' and v', is connected with D_{2R} giving o_1 as before; if D_{2R} is likewise inaccessible the line v_4D_{4R} may be used to locate o_1.

POINTS NEAR THE LEVEL OF THE HORIZON

(152) If as in the case of point E, Fig. 8, a point is near the level of the horizon, by the method above it is possible to get but a very unsatisfactory point of intersection for e_1. Knowing that the *perspective* of a *vertical line* is a *vertical line*, it may be imagined that E is *dropped* on a vertical line to F and the perspective of F satisfactorily located at f_1; a vertical line through f_1 intersects $e's'$ in e_1, the required perspective.

CIRCLES IN PERSPECTIVE

(153) If the *plane* of a *given circle* is *parallel* to the *picture plane* the *perspective* is a true *circle;* its radius may be found by finding the perspectives of the center and one point of the circumference. If the *plane* of the *circle* is *oblique* to the *picture plane* its *perspective* is an *ellipse.* The major and minor axes of this ellipse can be determined; however, for practical work this is unnecessary; a construction known as the 8 *point method* is more easily used and with perfect satisfaction.

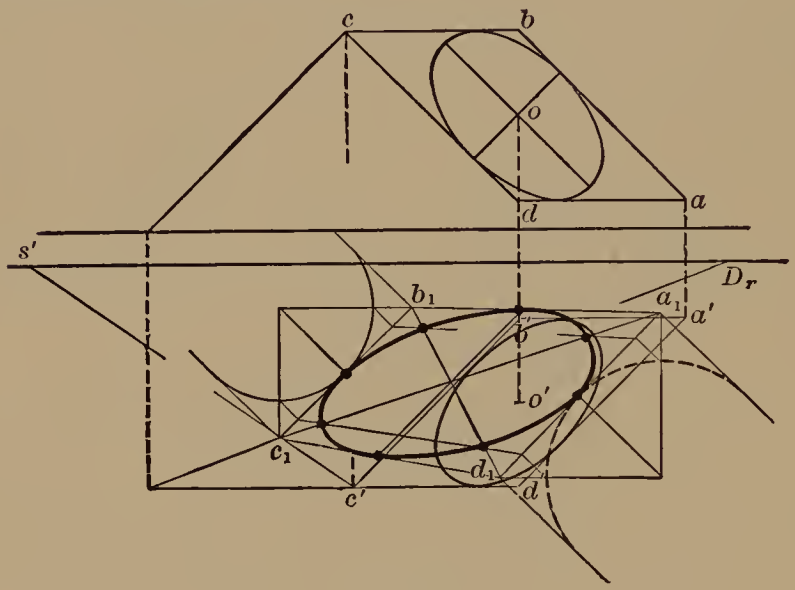


FIG. 9.

(154) Problem 45.—**To find by the eight point method the perspective of a given circle whose plane is oblique to the picture plane.**

Solution:

If a rhombus be circumscribed about the circle with two of its sides parallel to the picture plane (V) and two parallel to H, the perspective of this rhombus and its diagonals may be determined by methods already described. If then, with the perspective of one of the sides of the rhombus whieh is parallel to V, a square with its diagonals, center lines, and inscribed eircle, be constructed, the required eight points of intersection of these diagonals and

center lines with the circle, that is, eight points of the perspective of the given circle, are determined.

Construction:

Given: Circle with center O, Fig. 9.

Reqd: Perspective of circle.

The circumscribed rhombus is ABCD; the two sides AD and BC, Parallel to V, project on V in their true length; the sides AB and DC, parallel to H project on H in their true length. The perspective of the rhombus is $a_1b_1c_1d_1$, found by perpendiculars and diagonals. Since AD and BC are parallel to V their perspectives are parallel to their V projections. Then with a_1d_1 as a side the square is constructed and the diagonals, center lines and inscribed circle drawn. Inasmuch as the vanishing point of the lines a_1b_1 and c_1d_1 comes without the limits of the drawing, it is necessary to construct a second square on b_1c_1 to locate 4 of the 8 points. No further explanation is necessary as the construction is evident from the figure.

(155) Problem 46.—**To find the perspective of a given sphere.**

Discussion:

If with the point of sight as an apex a cone of tangent visual rays be drawn to the sphere, the line of tangency though not a meridian circle, will to the observer be the bounding line of the body and is known as the line of **apparent contour**. The perspective of this line of *contour* is the bounding line of the perspective.

Solution:

If with the point of sight as an apex a cone of visual rays be passed tangent to the sphere, the perspective of its line of tangency with the sphere (a circle), the apparent line of contour, is the required perspective of the given sphere.

Construction:

Given: Sphere O, Fig. 10.

Reqd: Perspective of the sphere.

The point of sight is connected with O, the center of the sphere. Revolving S about the vertical axis of the sphere until SO is parallel to V, s' moves into the position s'_r and the projection of the revolved cone may be drawn. $c'_r c'_r$ is the V projection of the line of apparent contour in this revolved position and equal to the diameter of the circle. Angle $o's'_r s'$ equals angle " θ " of the axis of the cone, hence, projecting $c'_r c'_r$ upon a horizontal line through c'_r we obtain $c_2 c'_r$, the minor axis of the elliptical H projection of the circle. x is the center of this ellipse; the axes of the ellipse may then

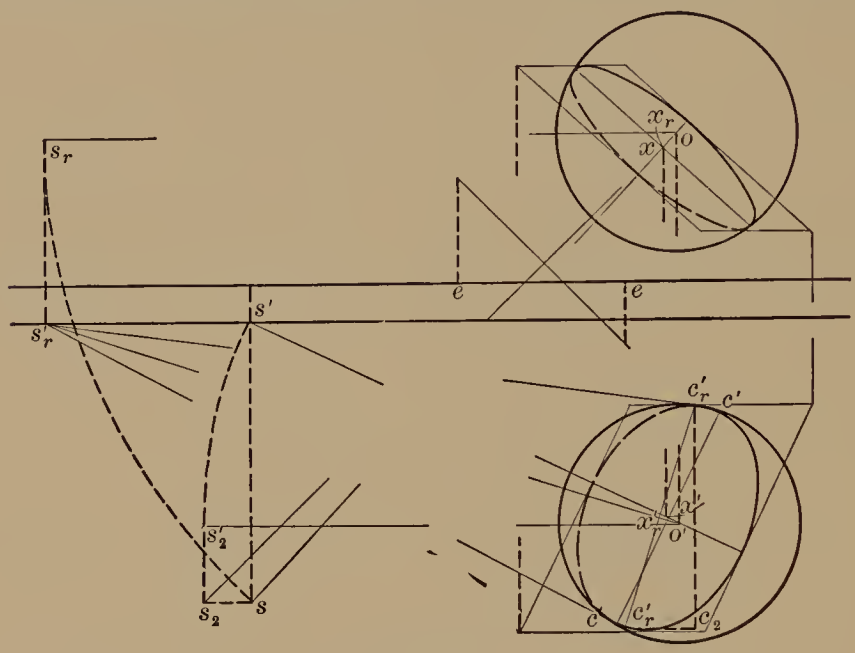


FIG. 10.

be drawn through x as follows: the major, perpendicular to so and equal to $c'_r c'_r$, the minor, coinciding with so and equal to $c'_2 c'_r$. The major axis of the elliptical V projection of the circular line of contour may be drawn through x' perpendicular to $s'o'$ and equal to $c'_r c'_r$. To determine the minor axis, SO is revolved about a perpendicular V through O until it is parallel to H. The angle between $s_2 o$ and GL equals angle " Φ " of the axis of the cone, hence a perpendicular to $s_2 o$, equal in length to $c'_r c'_r$, when projected on

GL gives *ee*, the required minor axis. Measuring along $s'o'$ on either side of x' one half of *ee*, the minor axis is located.

To find the perspective of the line of contour as represented by these ellipses, the 8 point method of Problem 51 can be used. In circumscribing the projections of the rhombus it is seen that in both projections one pair of sides is parallel to the major axis and perpendicular to the minor.

The perspective of the sphere will be an ellipse here which is contrary to natural observation since in looking at a sphere it always appears circular. The reason for this apparent inconsistency is due to the fact that the perspective is found where the lines of sight pierce a picture **plane;** and since the tangent visual cone mentioned above is a right circular cone with its axis (line SO) oblique to the picture plane, the piercing points of all the visual rays will form a perspective of the sphere which is an ellipse.

The usual use of the picture plane is seldom strictly in agreement with nature; to agree absolutely with natural observation, we should find the perspective not on a picture plane but on a **spherical surface** with its center at the point of sight. Due to the complications encountered in finding the piercing points of the lines of sight with a sphere, a **spherical perspective** is seldom used.

SHADOWS IN PERSPECTIVE

(156) Before taking up any problems in finding the perspectives of shadows it may be well to mention a few facts of the principles of shadows:

1. The *shadow* of a *given point* on a *given surface* is the *piercing point* in that *surface* of a *shadow ray* through the *given point.*

2. The *shadow* of a *given point* on a *given plane* is also the *intersection* of the *shadow ray* with the *projection* of the *shadow ray* on the *given plane.*

3. The *vanishing point* of *rays* of *light* may be *determined*

exactly as the *vanishing point* of any *other group* of *parallel lines*. See V, Fig. 11.

4. Since it is permissible to *assume* the *direction* of *light* the draftsman has a perfect right to *assume* at will the *vanishing point* of *rays*. This has the added advantage of placing the shadow more nearly where it is wanted. See Fig. 12.

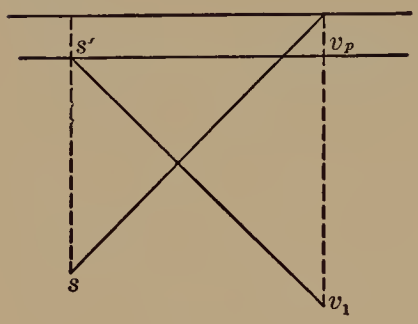


FIG. 11.

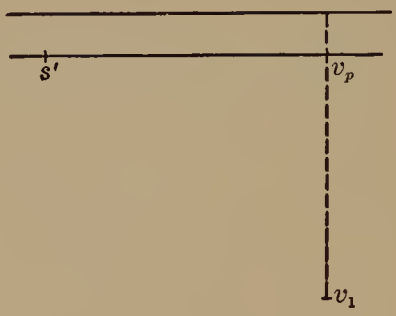


FIG. 12.

5. Shadows are most frequently desired on horizontal planes. The projections of rays on horizontal planes are horizontal lines, hence have their vanishing point in the horizon. Since the *ray* and its *projection* lie in the *same vertical plane* the *vanishing points* of *rays* and *projections* are on the *same vertical line*; hence on the location of the vanishing point of rays the vanishing point of projections on horizontal planes is immediately determined as the intersection with the horizon of a perpendicular to GL from V_1. See Fig. 12.

(157) Problem 47.—**To find the perspective of the shadow of a given point on a given horizontal plane.**

Solution:

The intersection of the perspective of the shadow ray from the given point and the perspective of the projection of this shadow ray on the given horizontal plane, is the required perspective of the shadow of the given point.

Construction:

Given: Point O and plane T, Fig. 13.

Reqd: Perspective of shadow of O on T.

The vanishing point of rays of light is assumed at v_1 the vanishing point of horizontal projections is located at v_p. By a perpendicular and a diagonal the perspective of O is o_1; o_1v_1 is then the perspective of the shadow ray from O. The projection of O on T is O_t, V projection o'_t, H projection coinciding with o. The perspective of O_t is o_t, and the perspective of the projection, o_tv_p. The required perspective of the shadow of O on T is then o_s, the intersection of o_1v_1 and o_tv_p.

(158) Problem 48.—**To find the perspective of the shadow of a given point on any oblique plane.**

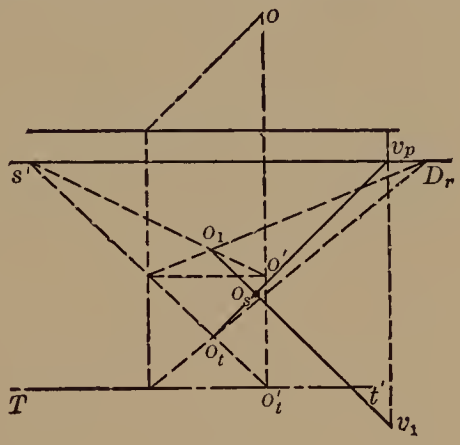


FIG. 13.

Solution:

The intersection of the perspective of the shadow ray from the given point with the perspective of the projection of this shadow ray on the given oblique plane, is the required perspective of the shadow of the given point.

(159) Problem 49.—**To find the perspective of the shadow of a given point on a given surface.**

Solution:

The intersection of the perspective of the shadow ray from the given point with the perspective of line cut from the surface by a plane through the shadow ray, is the required perspective of the shadow of the given point.

Construction:

Given: Point and cone X, Fig. 14.

Reqd: Perspective of shadow of O on X.

The assumed projections of a ray of light through S locate the vanishing point of rays of light at v_1. Through O pass a shadow ray, also a line to the apex X. These lines pierce the plane of the base of the cone in points P and P. The plane of the two lines cuts from the plane of the base

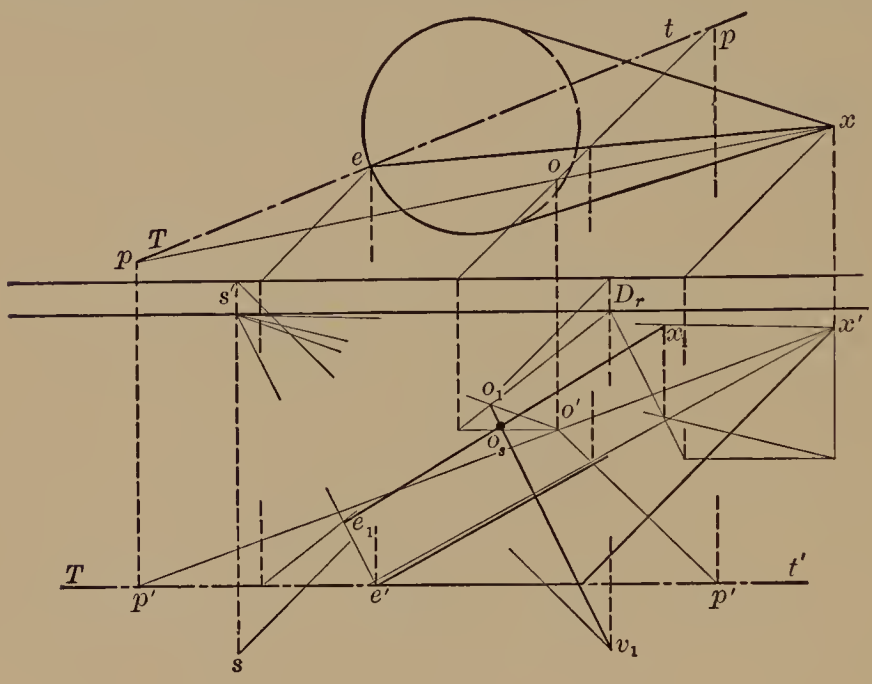


Fig. 14.

the trace *tt* and from the surface of the cone the element EX. The perspective of the element EX is e_1x_1, and the perspective of the shadow ray through O is o_1v_1. These perspectives intersect at o_s, the required perspective of the shadow of O on the given cone X

POSITION OF POINT OF SIGHT

(160) Though no fixed rules for placing the point of sight will be given, a few suggestions will be helpful.

With beginners there is a tendency to simplify construc-

tion by placing the point of sight *too close* to the object, perhaps within the limits of the drawing. The result is invariably a " steep " perspective having a most unnatural appearance. *Too great* a *distance* produces *always less serious faults* than *too small* a distance. With large objects the point of sight should be before the picture plane a distance of at least three or four times the greatest dimension of the object.

More natural perspectives are produced with the point of sight nearly in *front* of the object. With the point too far to the side trouble is likewise experienced with too long intersections.

With small objects any rule but that of " common sense " is hard to give. The best solution of the matter is to ask one's self the question: If this object were before me, how far from it should the eye be, and in just what position to see all of the details to the best advantage? A few minutes, thought usually suffices to answer the question satisfactorily.

PART IV

RECITATION EXERCISES

The following exercises are given for the convenience of both instructor and student in assigning problems for oral recitation. In connection with each set of problems a number of requirements are given which may be made of each problem.

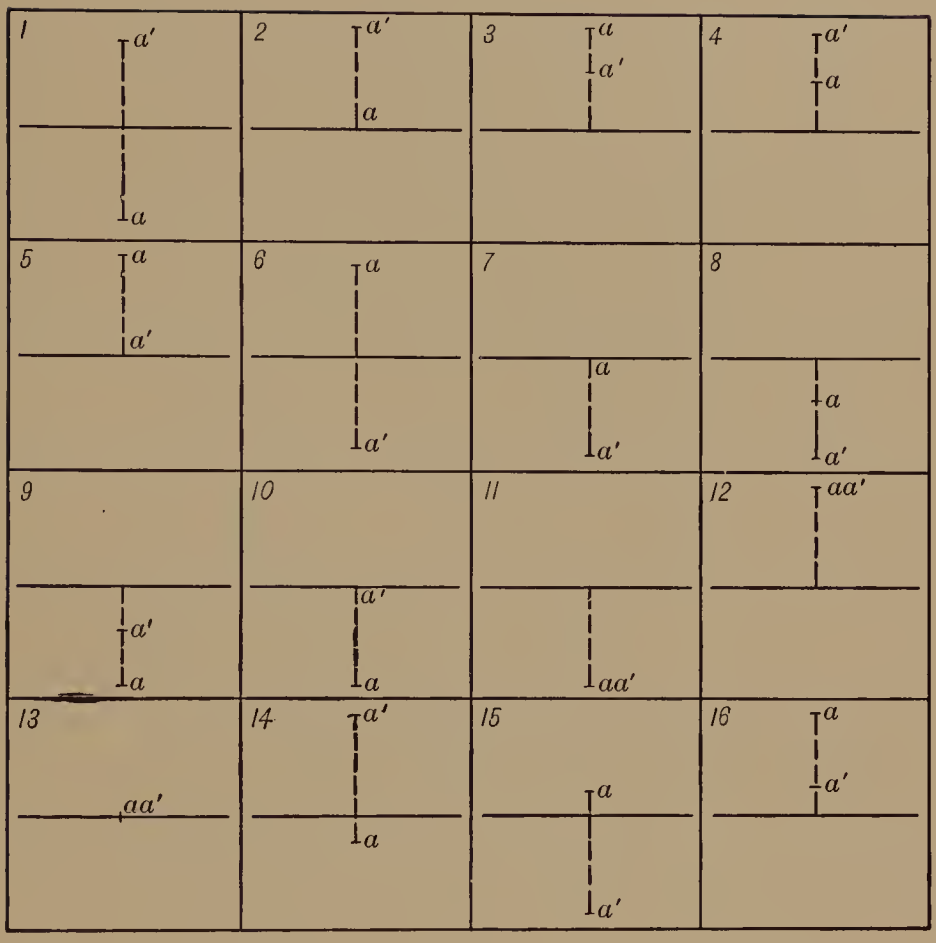
It is suggested that the problems be assigned numerically as follows: 1—6—2; this assignment reads Exercise 1, Problem 6, to be solved for Requirement 2.

EXERCISE No. 1

Requirements

1	Required:	A line through point A.
2	"	A line from point A passing through ? quadrant into ? quadrant.
3	"	A line through point A, Z″ long and at angle " θ " or " Φ " with H or V.
4	"	Two lines through point A at 90° to each other.
5	"	A plane through point A.
6	"	A plane through point A at angle " θ " or " Φ " with H or V.
7	"	A point on GL, Z″ from point A.
8	"	Five (5) points in space each Z″ from point A.
9	"	A line through point A at given angles " θ " and " Φ " with H and V.
10	"	A plane through point A at given angles " θ " and " Φ " with H and V.

EXERCISE 1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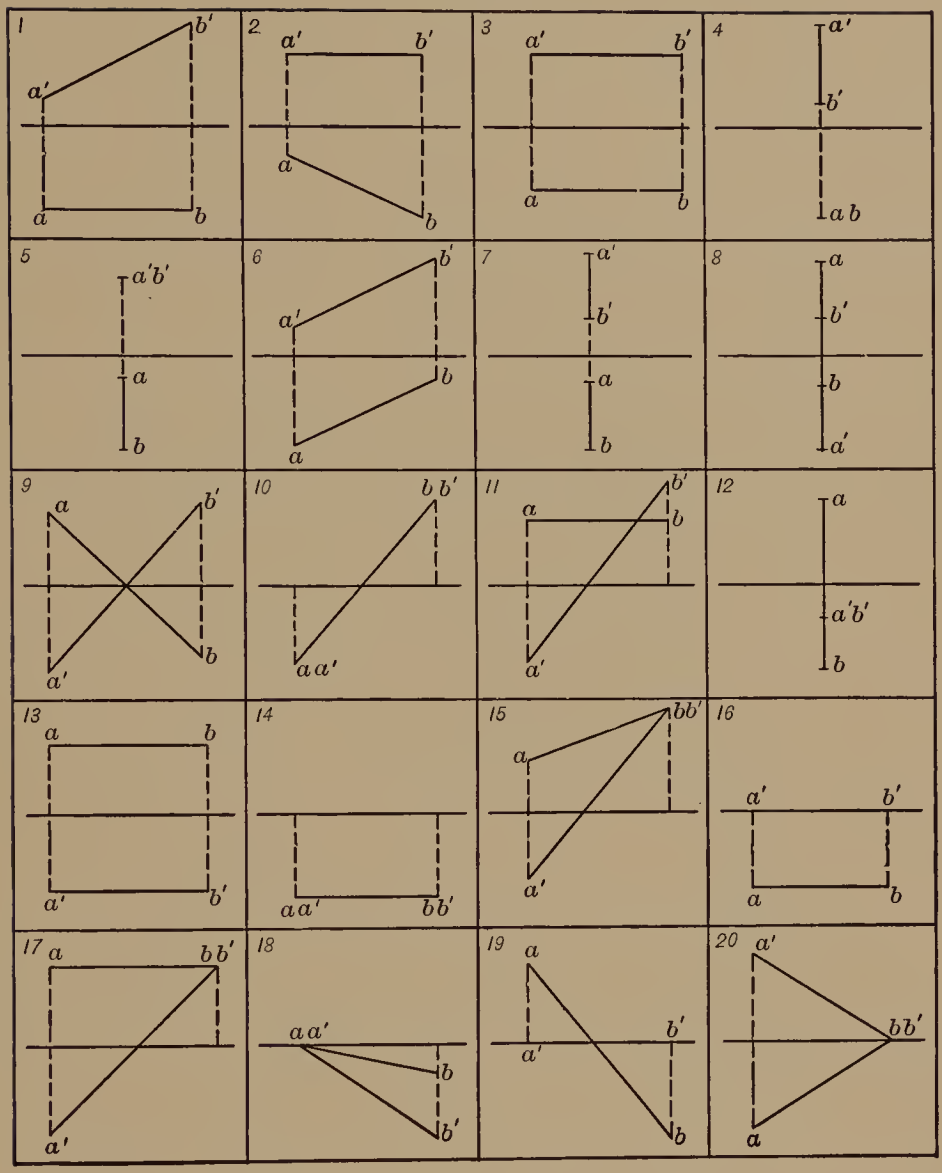
EXERCISE NO. 2

Requirements

1	Required:	Two (2) points, each Z″ from line AB.
2	"	A line BC making 90° with AB.
3	"	A plane through A perpendicular to line AB.
4	"	True length of line AB.
5	"	A line through A at angle " X " with line AB.
6	"	A plane through A at angle " X " with line AB.
7	"	Angles " θ " and " Φ " of line AB.
8	"	A plane through AB at given angle " O " or " Φ " with H or V. (Given angle equal to or greater than corresponding angle of the given line.)
9	"	A point on GL, Z″ from line AB.
10	"	P projection of AB.
11	"	H, V, and P piercing points of AB.

EXERCISE 2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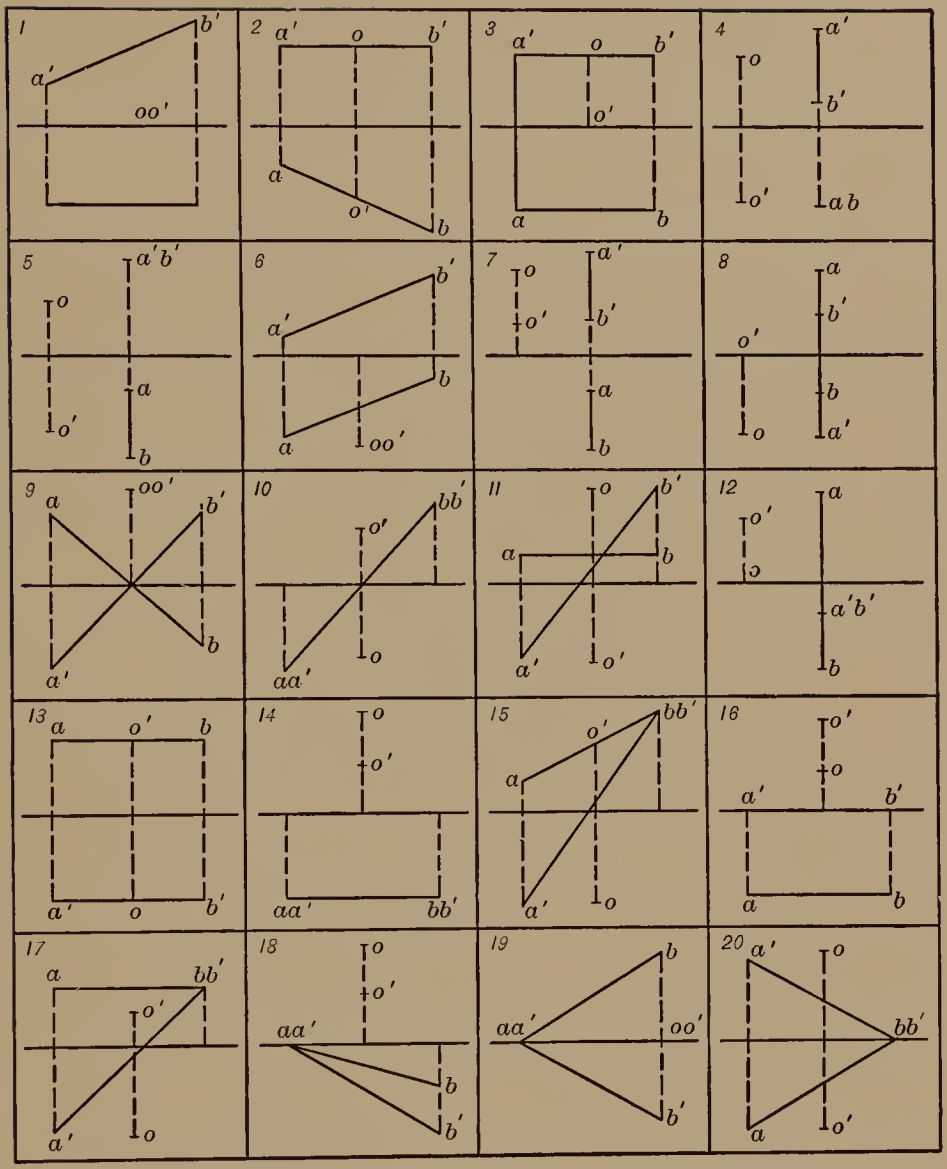
EXERCISE NO. 3

Requirements

1 Required: A plane through point O parallel to line AB.
2 " A plane through point O and line AB.
3 " A plane through point O perpendicular to line AB.
4 " The distance from point O to line AB.
5 " A line through point O at given angle " X " with line AB.
6 " A plane through point O at given angle " X " with line AB.

EXERCISE 3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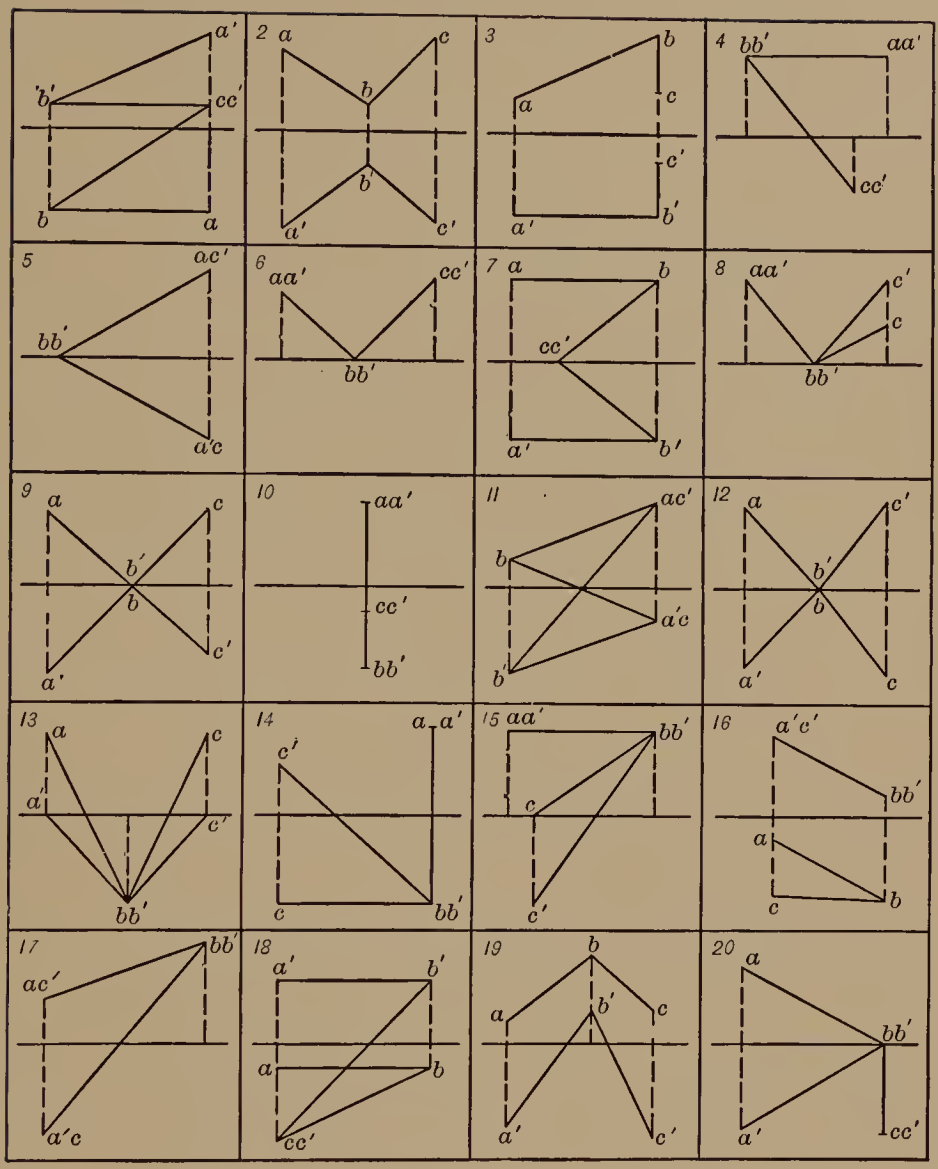
EXERCISE NO. 4

Requirements

1 Required: A plane through lines AB and BC.
2 " True value of angle " X."
3 " Projections of the bisector of angle " X."
4 " Locus of points equidistant from lines AB and BC.

EXERCISE 4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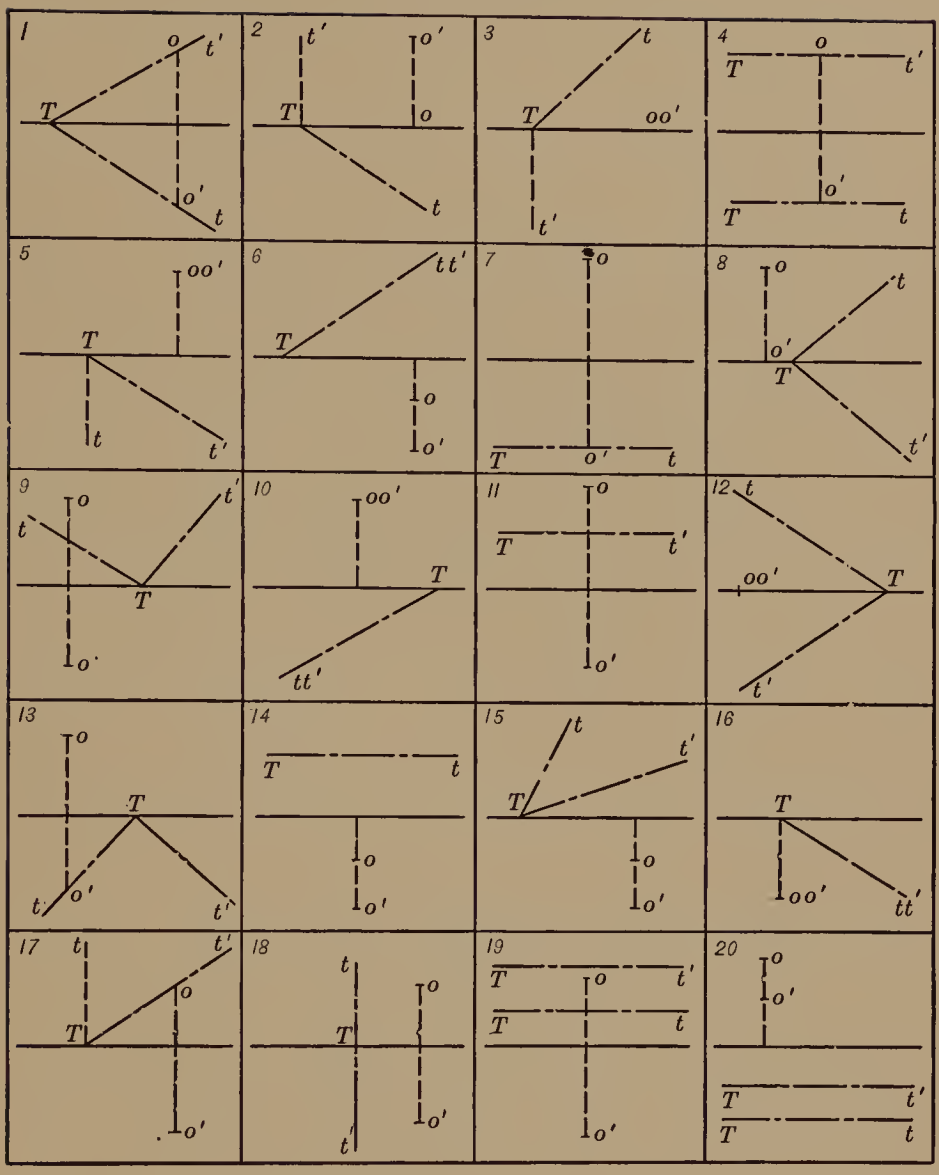
8 aa' c' c bb'
9 a c b' b c' a'
10 aa' cc' bb'
11 ac' b a'c b'
12 a c' b' b a' c
13 a c a' c' bb'
14 a a' c' c bb'
15 aa' bb' c c'
16 a'c' bb' a c b
17 bb' ac' a'c
18 a' b' a b cc'
19 b a b' c a' c'
20 a bb' a' cc'

EXERCISE NO. 5

Requirements

1	Required:	Line through point O parallel to plane T.
2	"	A plane through point O parallel to plane T.
3	"	The projection of point O on plane T.
4	"	The distance from point O to plane T.
5	"	A line through point O at given angle " X " with plane T.
6	"	A plane through point O at given angle " X " with plane T.
7	"	Angles " θ " and " Φ " of plane T.
8	"	A plane through Tt or Tt' at given angle " X " with plane T.
9	"	A point in plane T, Z″ from point O.

EXERCISE 5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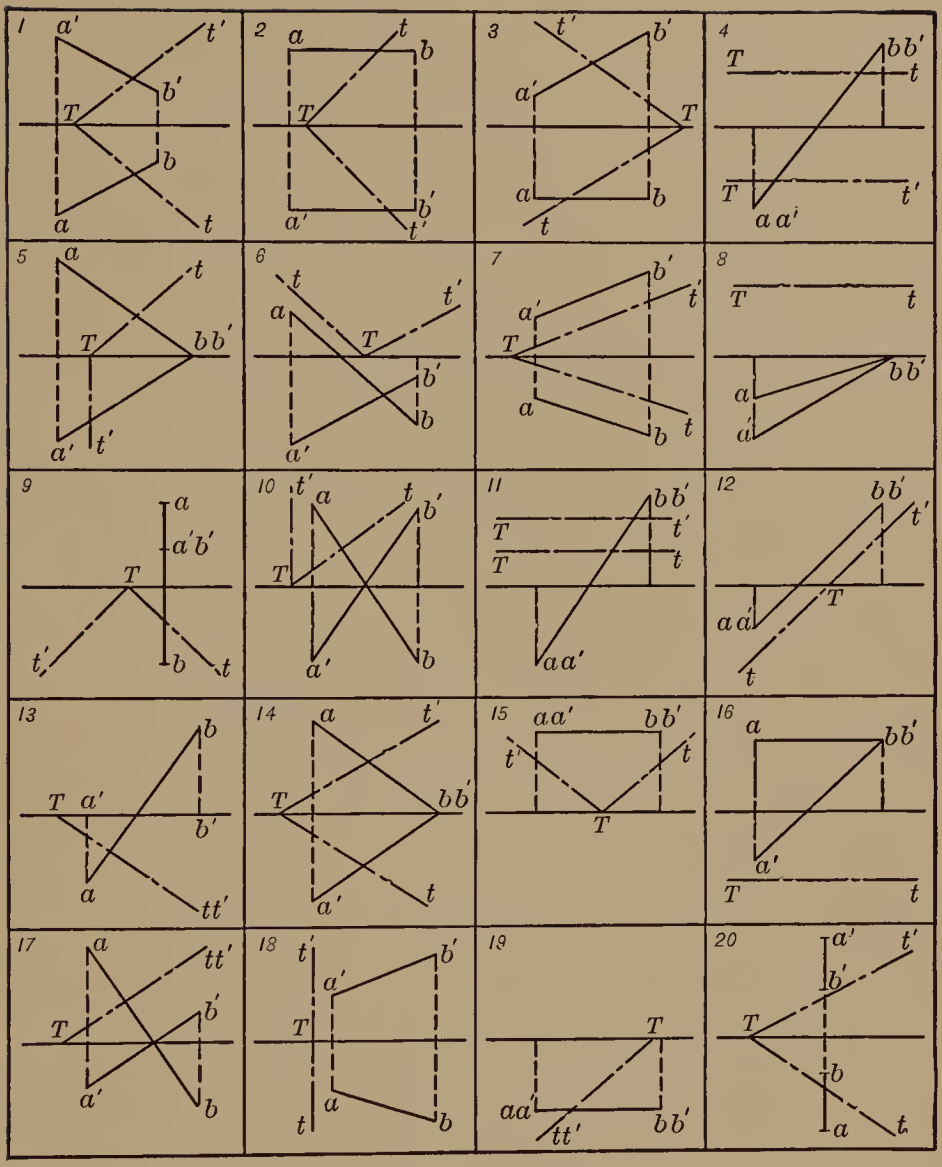
EXERCISE NO. 6

Requirements

1 Required: The piercing point of line AB in plane T.
2 " The projection of line AB on plane T.
3 " Angle " X."
4 " A line in plane T at given angle " X " with line AB. (Given angle " X " or equal to greater than angle " X " of line AB.)
5 " A plane through line AB perpendicular to plane T.

EXERCISE 6

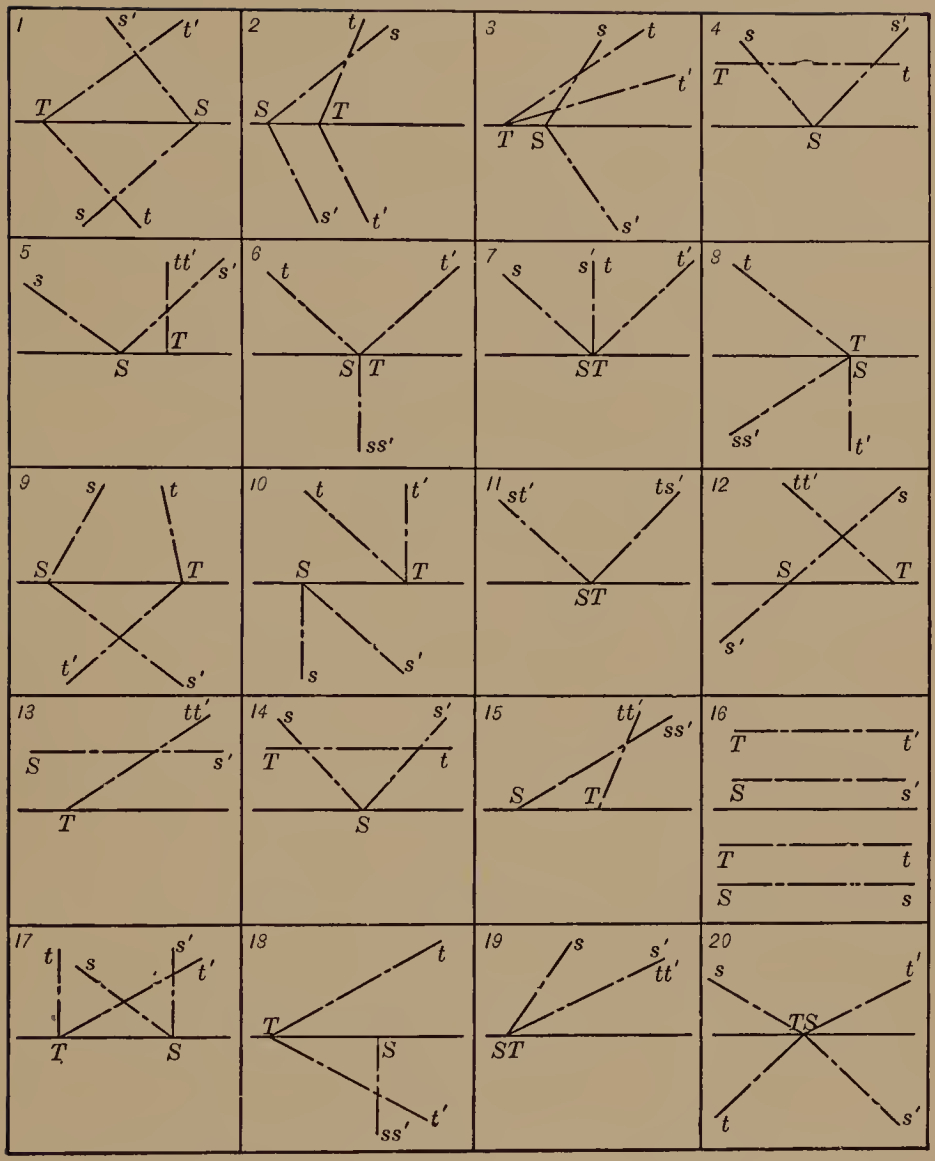


EXERCISE NO. 7

Requirements

1 Required: Line of intersection
2 " Angle " X."
3 " Plane bisecting angle " X."
4 " A plane perpendicular to S and T.

EXERCISE 7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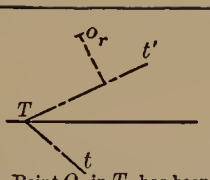
EXERCISE NO. 8

These are original problems, having individual requirements which accompany each.

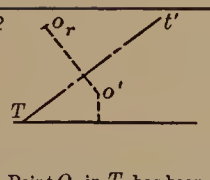
EXERCISE 8

1



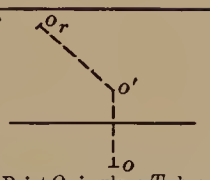
Point O, in T, has been revolved about Tt' into V at o_r. Find o' and o.

2



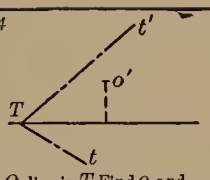
Point O, in T, has been revolved about Tt' into V at o_r. Find Tt and O.

3



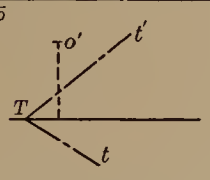
Point O, in plane T, has been revolved about Tt' into V at o_r. Find Tt' and Tt.

4



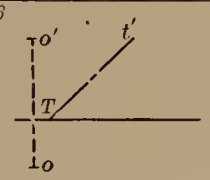
O lies in T. Find o, and the locus of points in T, at distance X from O.

5



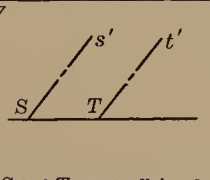
O is X'' from T. Find o.

6



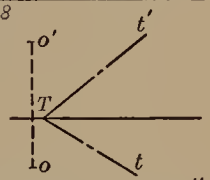
O is X'' from T. Find Tt.

7



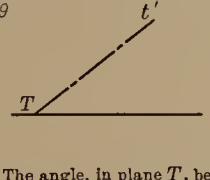
S and T are parallel and X'' apart. Find Ss and Tt.

8



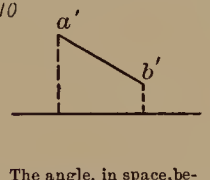
Find 3 points in T, X'' from O.

9



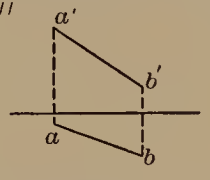
The angle, in plane T, between Tt' and Tt, is 60°. Find Tt.

10



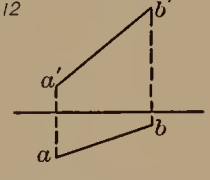
The angle, in space, between $a'b'$ and ab is 45°. Find ab.

11



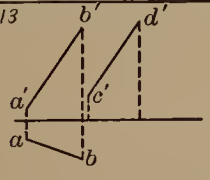
Find a point in H, which is X'' from AB.

12



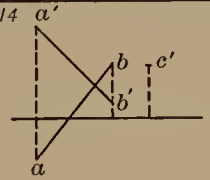
Find a line in H, which makes 60° with AB.

13



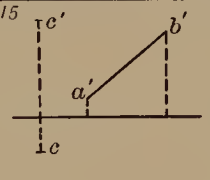
AB and CD are parallel and X'' apart. Find cd.

14



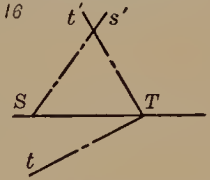
C is X'' from AB. Find c.

15



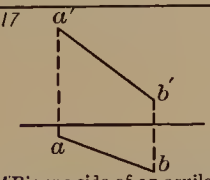
C is X'' from AB. Find ab.

16



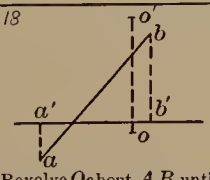
Angle $X = 90°$. Find Ss.

17



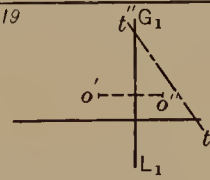
AB is one side of an equilateral $\triangle$; for the plane of the $\triangle$, $\Theta = 60°$. Draw the $\triangle$.

18



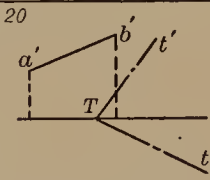
Revolve O about AB until it is in third quadrant, equidistant from H and V.

19



O lies in T. Find Tt', Tt and o.

20



Angle $X = 60°$. Find ab.

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